Charles
University

Faculty of Mathematics and Physics;

Departemnet of Mathematics Education

Developing conceptual knowledge in school mathematics

Lesson #1

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The aim of this course:

- Conceptual understanding
- Secondary school level
- Problem-solving
- Homework (borji@karlin.mff.cuni.cz) [Tuesday 12:00]

Topics of the course:

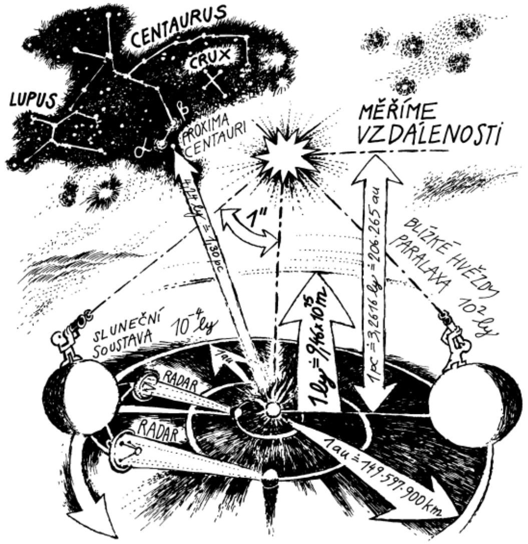
- Exponential and logarithmic functions
- Trigonometric functions and their inverses
- Derivatives
- Limits of sequences
- Series and their convergence
- Combinatorics

Interpretation of Exponential and Logarithmic Functions in Contextual Situations

Some real-world situations that we experience in everyday life, or some pseudorealistic situations.

Why are exponential and logarithmic functions important?

- They are fundamental topics in secondary school and university study programs.
- They are necessary for understanding more advanced mathematical concepts.
- Logarithmic scales are used to measure many real-world phenomena, such as the intensity of earthquakes, the intensity of sound, and the brightness of stars.



- If $x > 0$,

  the logarithm of x with base a ($a > 0, a \neq 1$),
  is a real number y
  such that $\boldsymbol{a^y = x}$.
  We write this as: $\mathbf{log}_{\boldsymbol{a}}(\boldsymbol{x})$. (Euler, 1770/1984)

Inverse

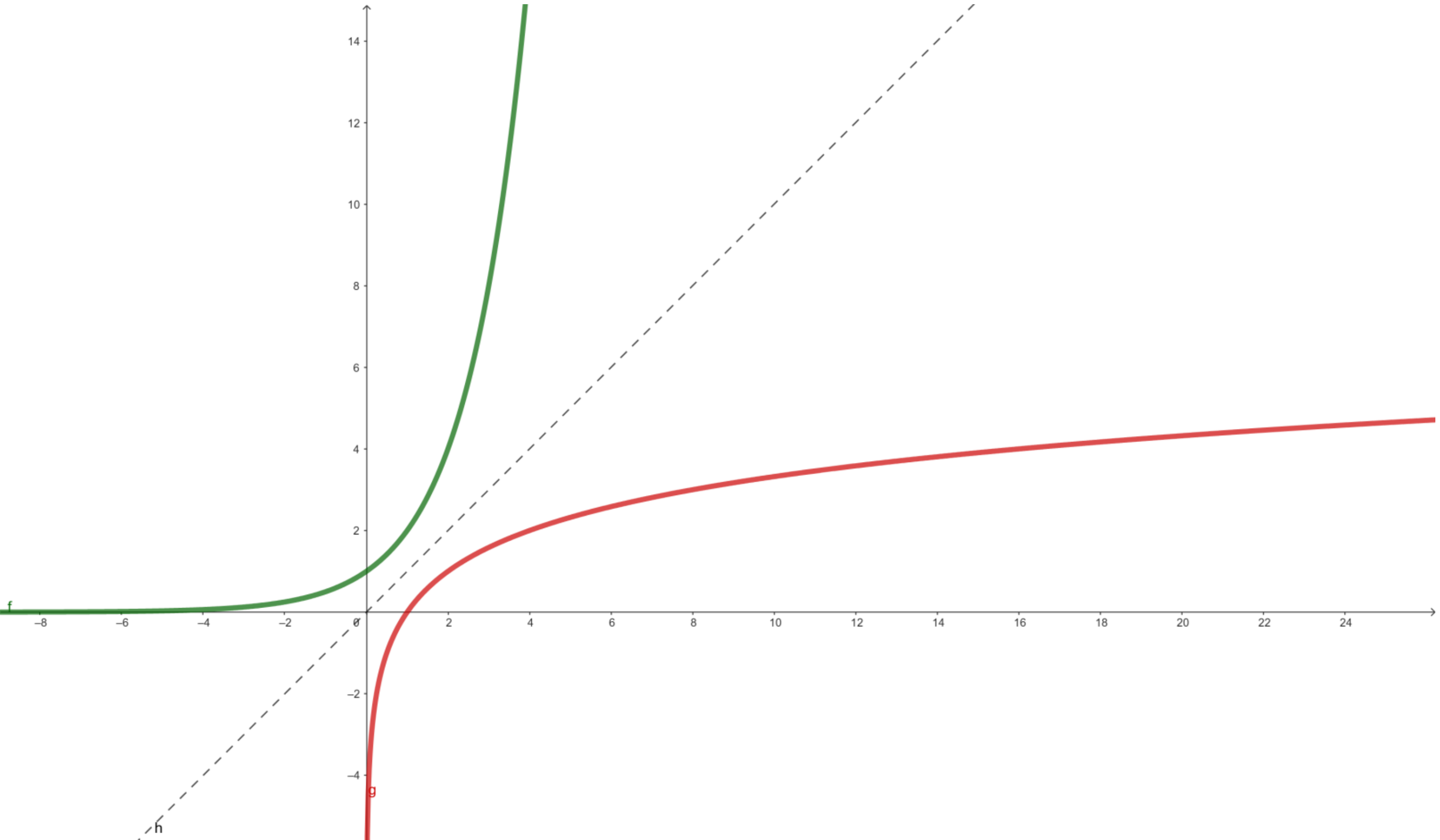
$$f = a^x; a \in R^+ \setminus \{1\} \Longleftrightarrow f^{-1} = \log_a x\,; a \in R^+ \setminus \{1\}$$

$$\log_3 9 = x \quad \Rightarrow 3^x = 9 \quad \Rightarrow x = 2$$

$$\log_5 5^5 + \log_5 \frac{1}{5} = ?$$

Draw the graph of $y = log_2 x$

$$\boldsymbol{f = 2^x} \overset{\text{Inverse}}{\Longleftrightarrow} \boldsymbol{f^{-1} = \log_2 x}$$



Logarithmic and exponential equations

$$2^x + 2^{x+1} = 24$$

$$\log_8 (x-1)^2 = 0$$

$$\log_{10} x + \log_{10} 3 = 2 \log_{10} 4 - \log_{10} 2$$

$$\log_{10} x + \log_{10} 3 = 2\log_{10} 4 - \log_{10} 2$$

$$log_{10} 3x = log_{10} 16 - log_{10} 2$$

$$log_{10} 3x = log_{10} 8$$

$$3x = 8$$

$$x = \frac{8}{3}$$

What does each step mean? What does each logarithm and the unknown x?

What problems exist for teachers and students in teaching and learning exponential and logarithmic functions?

- Most students and some teachers are not able to interpret the concepts of exponent and logarithm in real or realistic situations. This is an international problem.
- Students know that $\log_2 8$ is 3, but most of them, and even some teachers, cannot explain this in a contextual situation.

Students often ask their mathematics teacher:

“When or where will I use these mathematical concepts in my life?”

$2^a \times 2^b = 2^{a+b}$

$\log_c a - \log_c b = \log_c \frac{a}{b}$

$\log_c a + \log_c b = \log_c a.b$

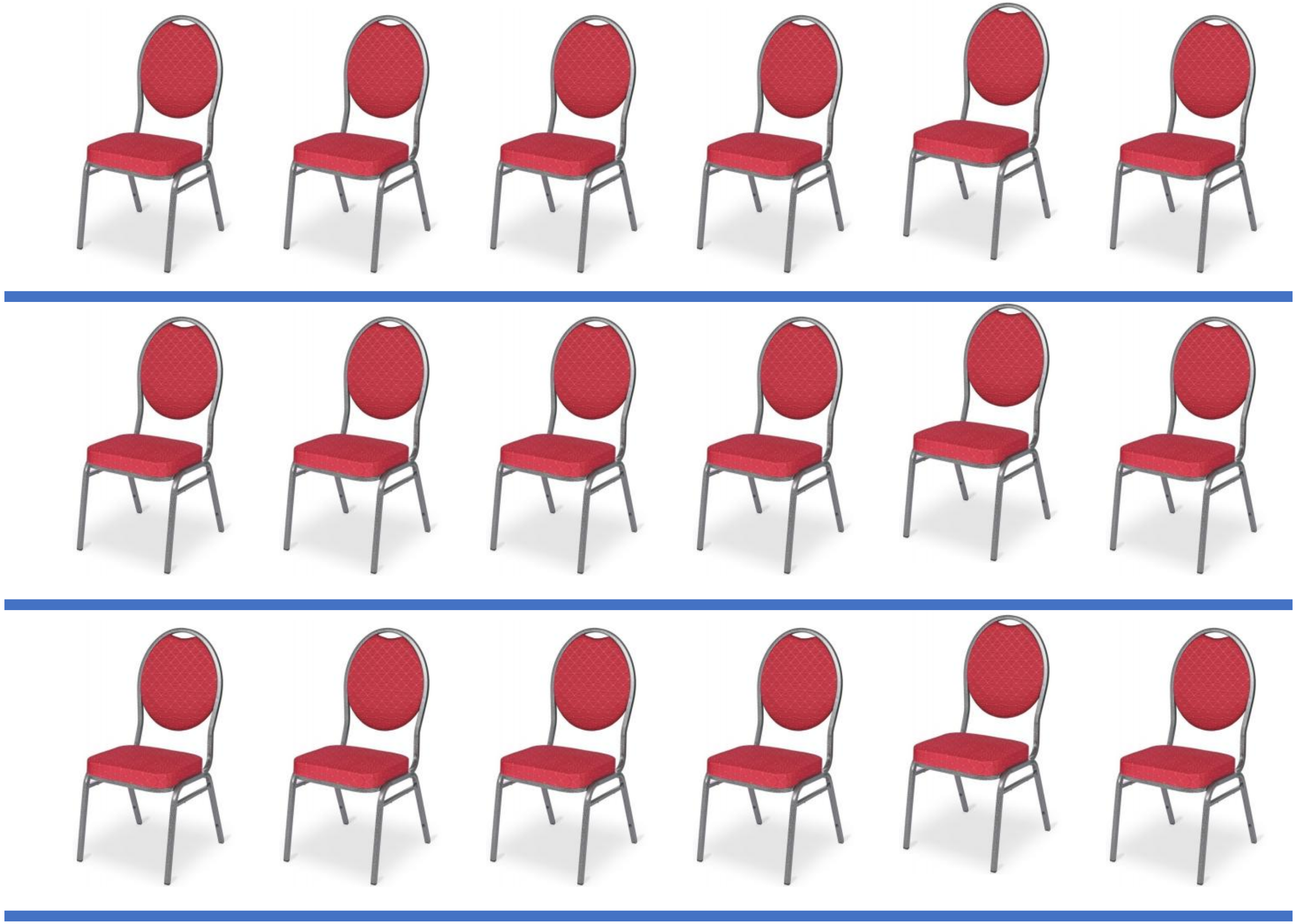
$\frac{\log_c a}{\log_c b} = \log_b a$

Elementary school students learn to create word problems for equations like $3 + 4 = 7$ nebo $2 \times 3 = 6$.

Yesterday I read 3 pages of my book. Today I read 4 pages. How many pages did I read in total?

$$3 + 4 = 7$$

$$\mathbf{3 \times 6 = 18}$$



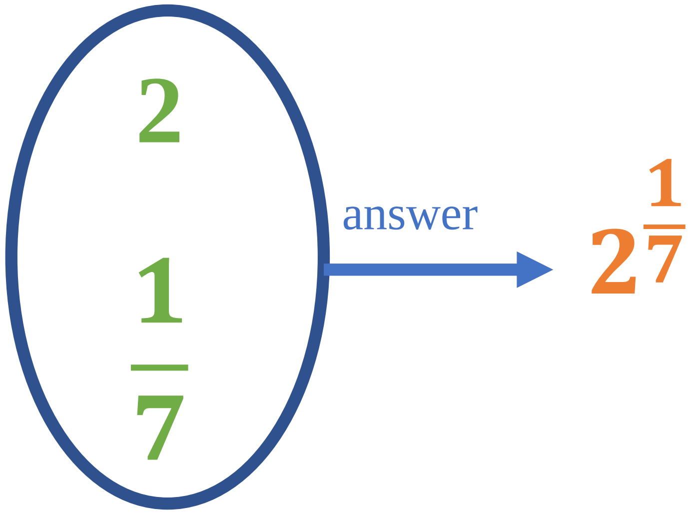
But can secondary school or university students interpret exponential and logarithmic functions in contextual situations?

$$\mathbf{2^{\frac{1}{7}}}$$

$$\log_2 \mathbf{3}$$

$$\log_2 \mathbf{7} + \log_2 \mathbf{4} = \log_2 \mathbf{28}$$

Design a word problem using the numbers **2** and $\frac{1}{7}$ such that the answer is $2\frac{1}{7}$.



2

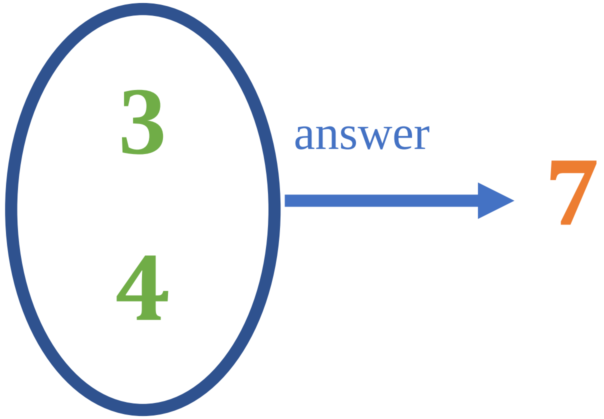
$\frac{1}{7}$

answer

$2\frac{1}{7}$

The word problem must be understandable for secondary school students.

Design a word problem using the numbers **3** and **4** such that the answer is **7**.



3

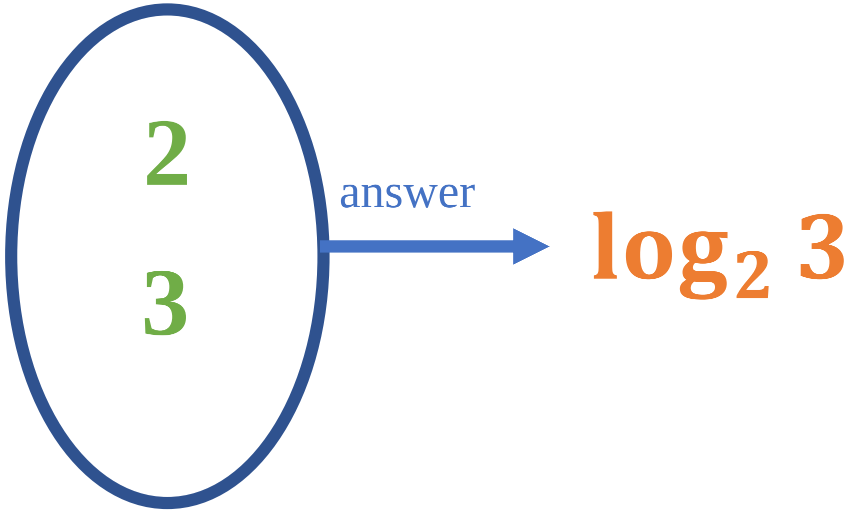
4

answer

7

Yesterday I read **3** pages of my book. Today I read **4** pages. How many pages did I read in total? $\boxed{7}$

Design a word problem using the numbers **2** and **3** such that the answer is **$\log_2 3$**.



2

3

answer

$\log_2 3$

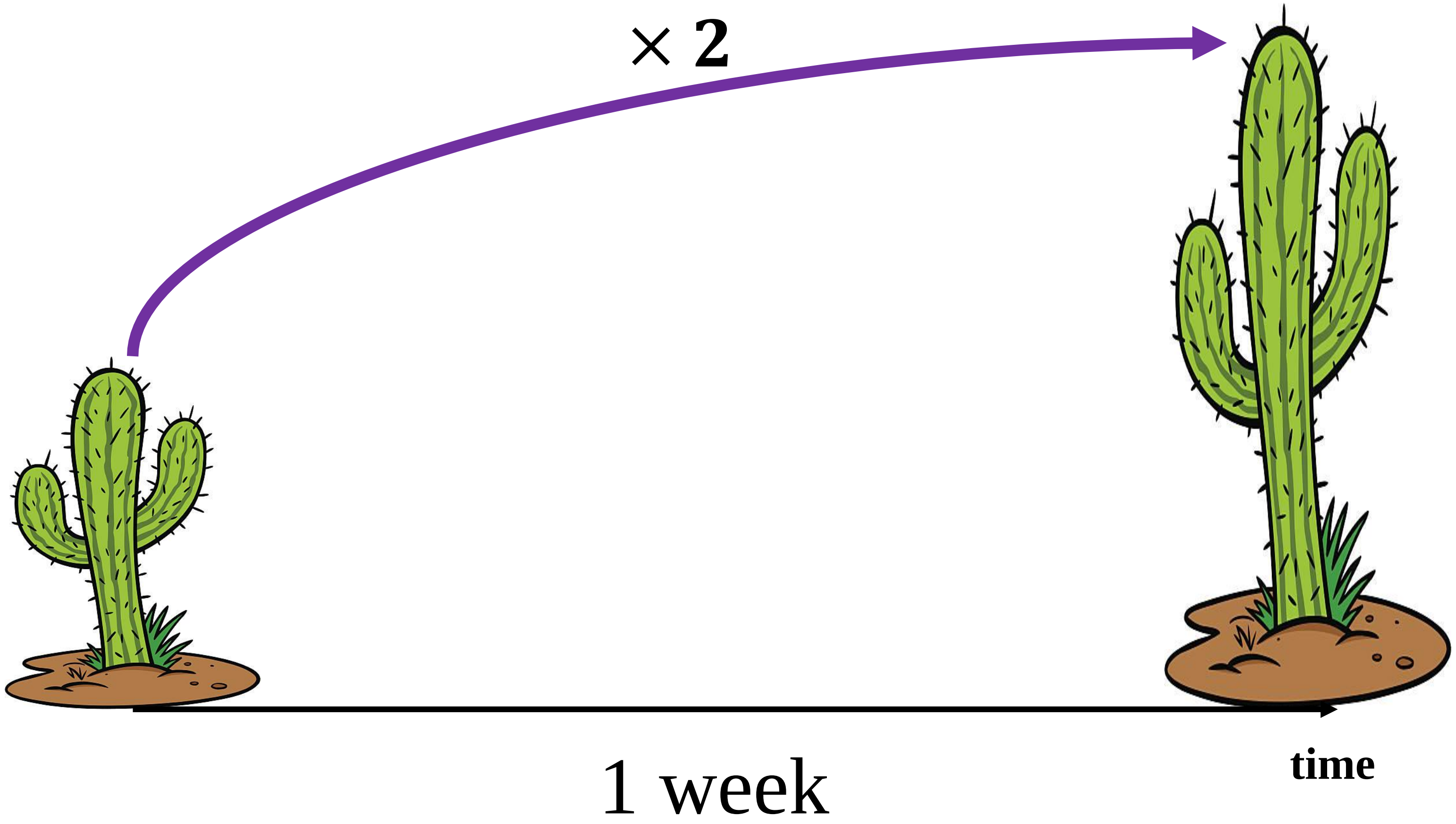
The word problem must be understandable for secondary school students.

Goal: How can we help students and teachers understand the concepts of the exponent and the logarithm and their rules in real-life situations?

$$\mathbf{2}^{\frac{\mathbf{1}}{7}} \qquad \log_2 \mathbf{3}$$

$$\log_2 \mathbf{7} + \log_2 \mathbf{4} = \log_2 \mathbf{28}$$

We use a story about a cactus.



× **2**

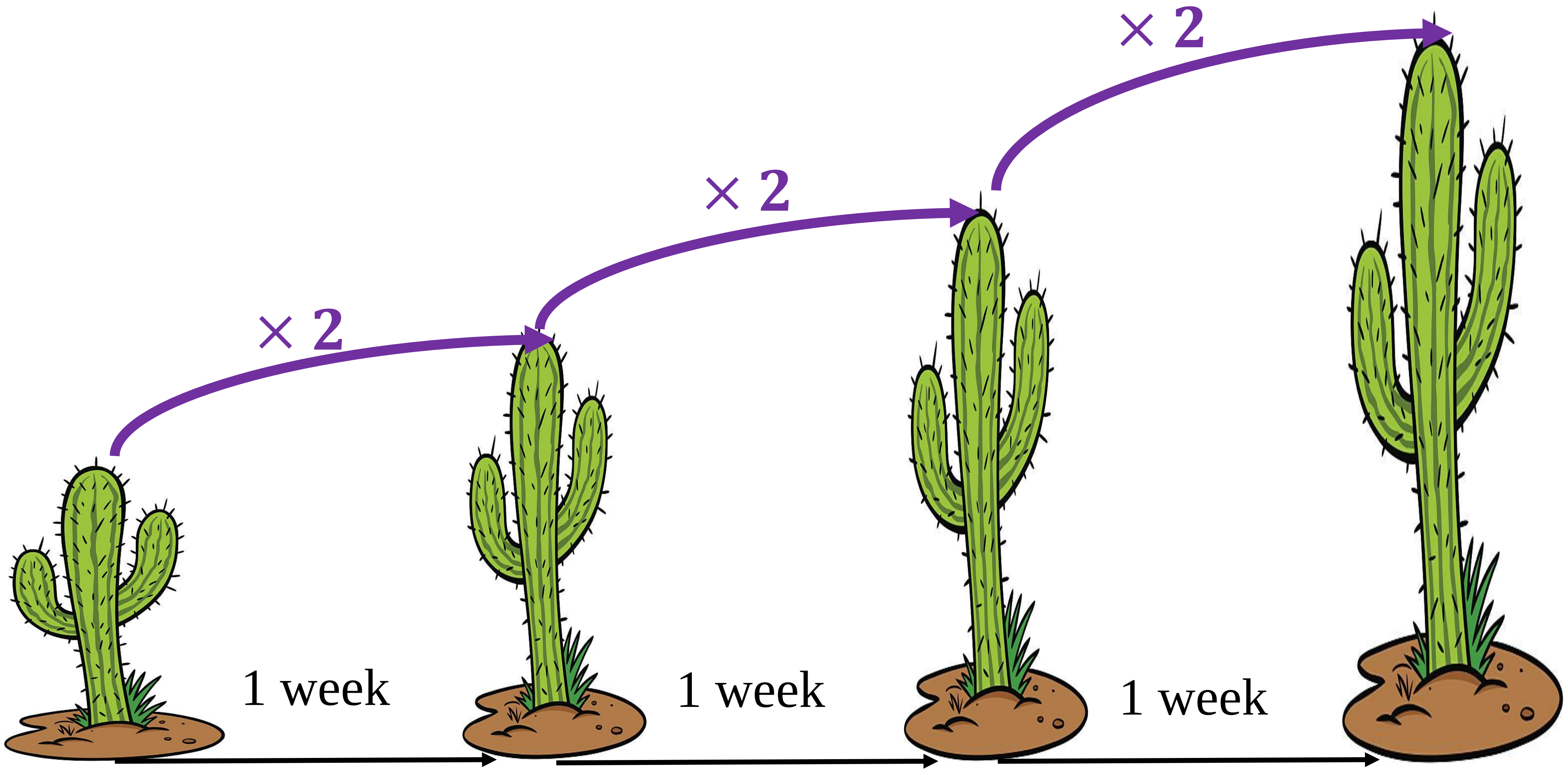
1 week

time

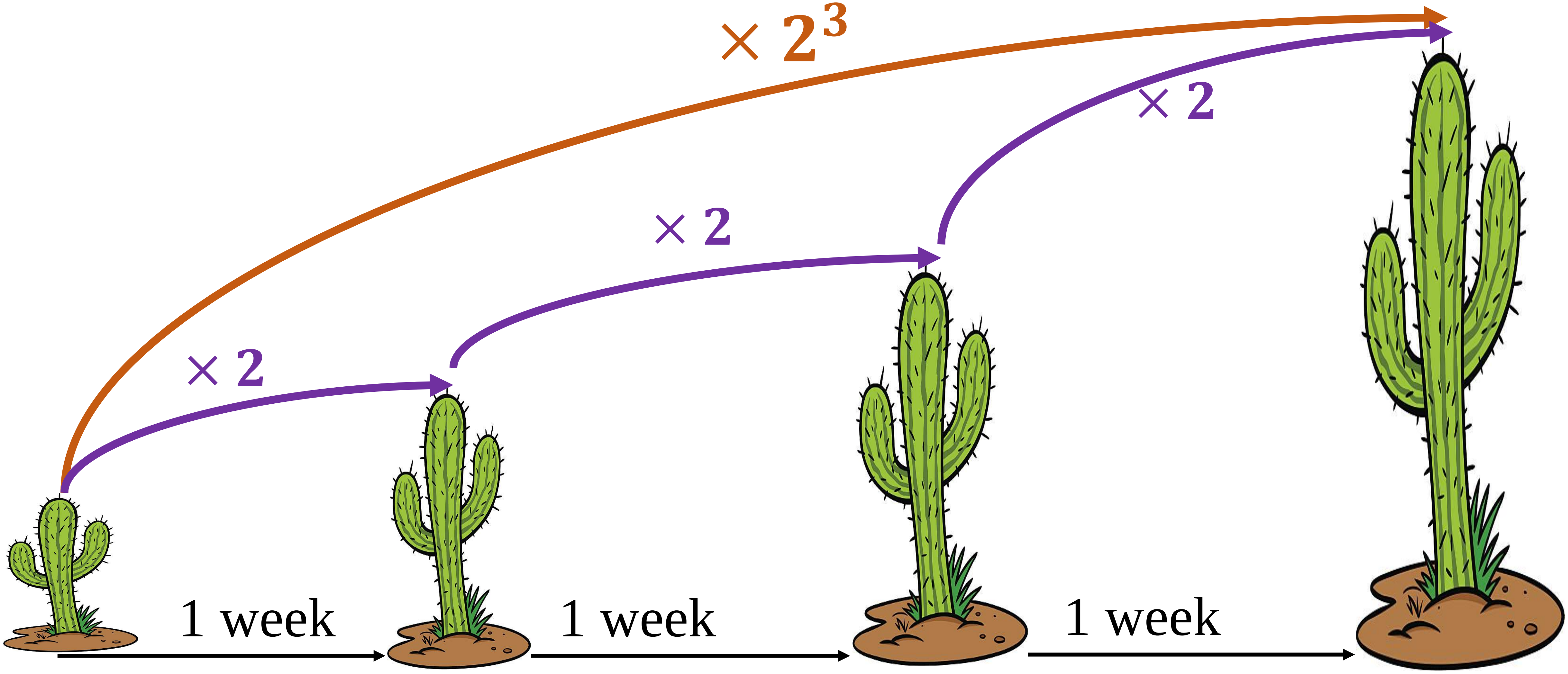
There are two quantities in this story:

1. Height of the cactus
2. Time

There is an exponential relationship between “height” and “time”



What does $\mathbf{2^3}$ mean in this story?



× 2^3

× 2

× 2

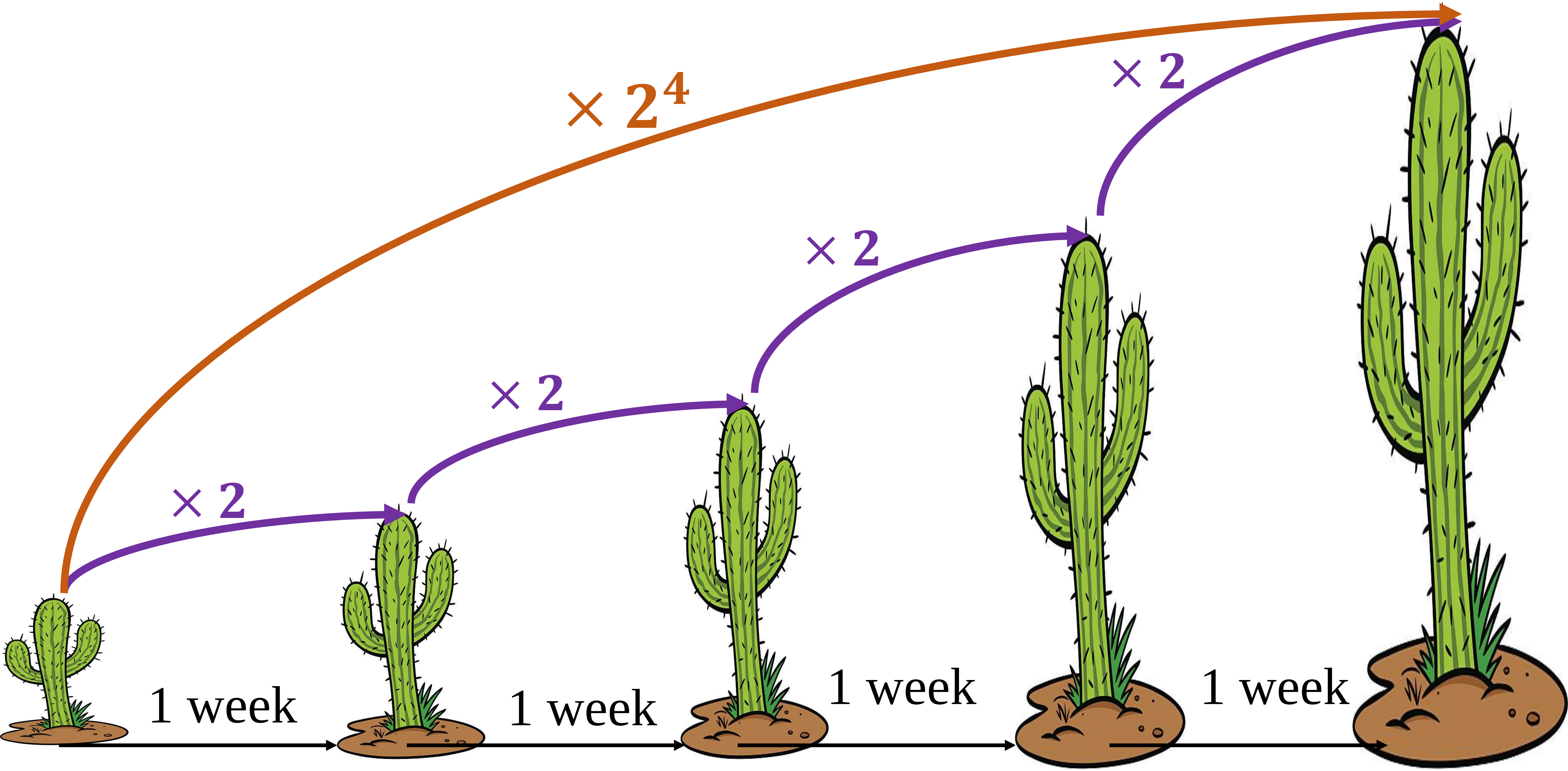
× 2

1 week

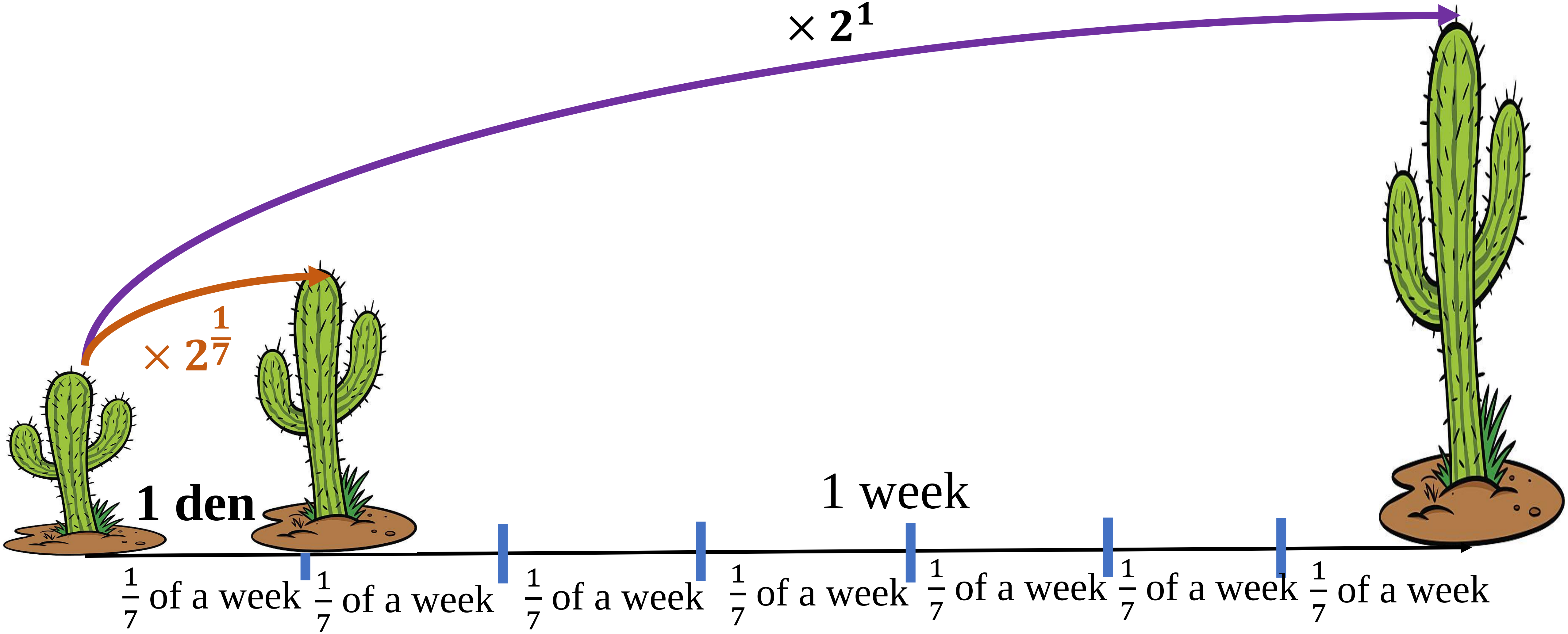
1 week

1 week

What does $\mathbf{2}^4$ mean in this story?



What does $2^{\frac{1}{7}}$ mean in this story?



$\times 2^{1}$

$\times 2^{\frac{1}{7}}$

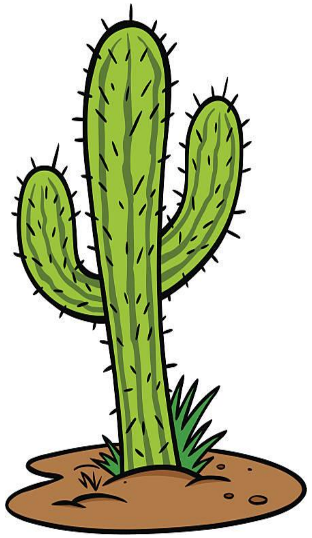
1 den

1 week

$\frac{1}{7}$ of a week $\frac{1}{7}$ of a week $\frac{1}{7}$ of a week $\frac{1}{7}$ of a week $\frac{1}{7}$ of a week $\frac{1}{7}$ of a week $\frac{1}{7}$ of a week

What does $\mathbf{2^0}$ mean in this story?

$$\mathbf{2^0 = 1}$$



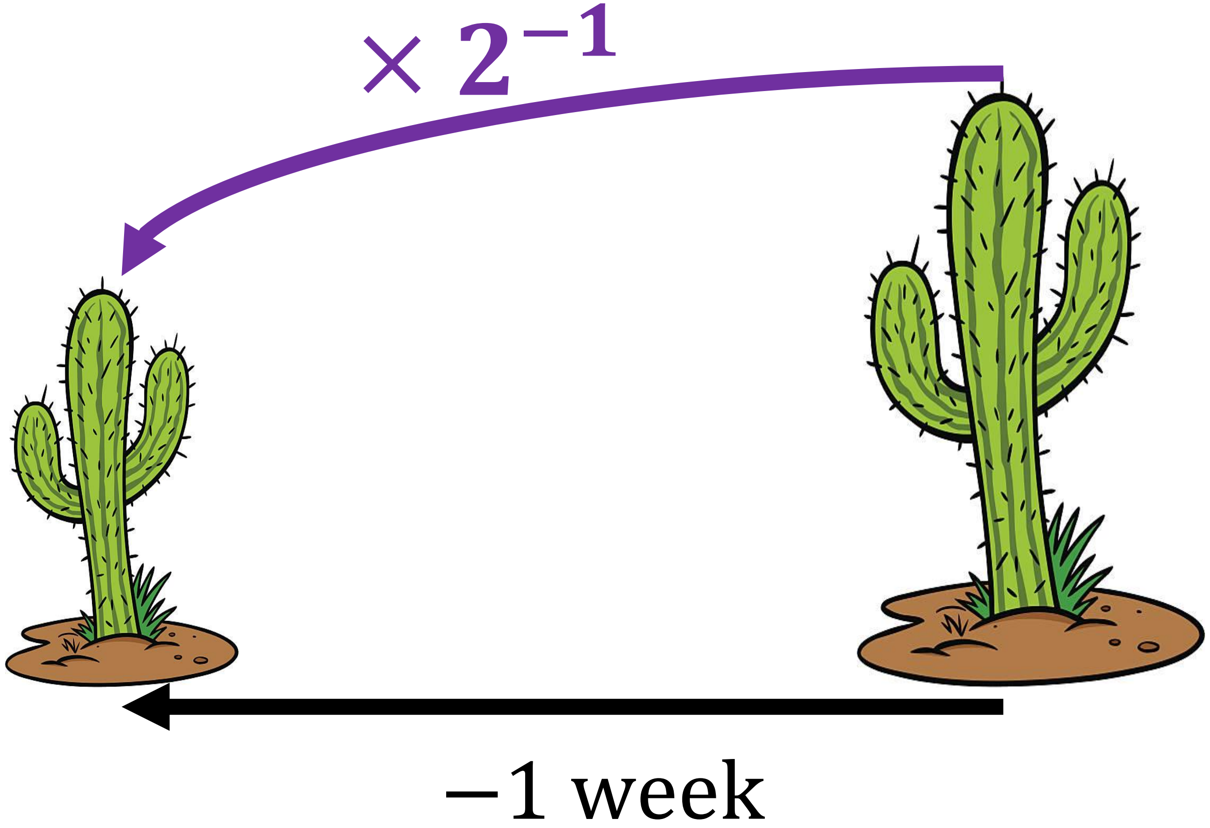
no time passes

$\mathbf{2^0}$

$=$

$\mathbf{1}$

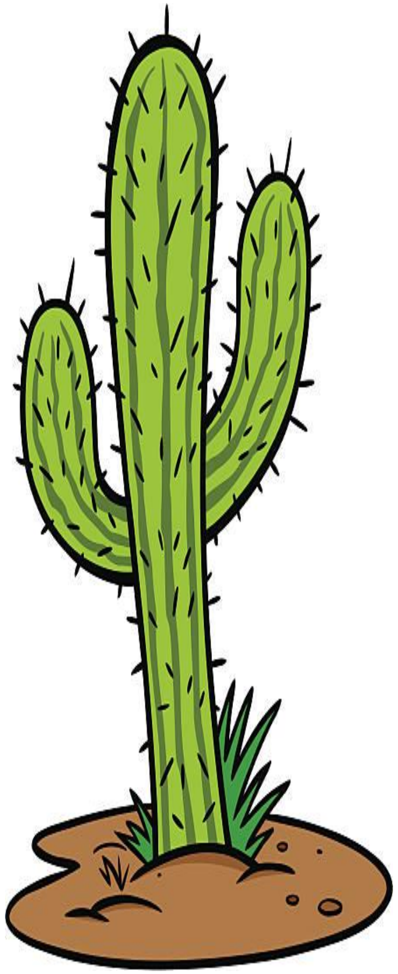
What does $\mathbf{2^{-1}}$ mean in this story?



$\times\ \mathbf{2^{-1}}$

-1 week

log



$$\log_2 8$$

How many factors of **2** are there in **8**?

$$\log_2 8$$

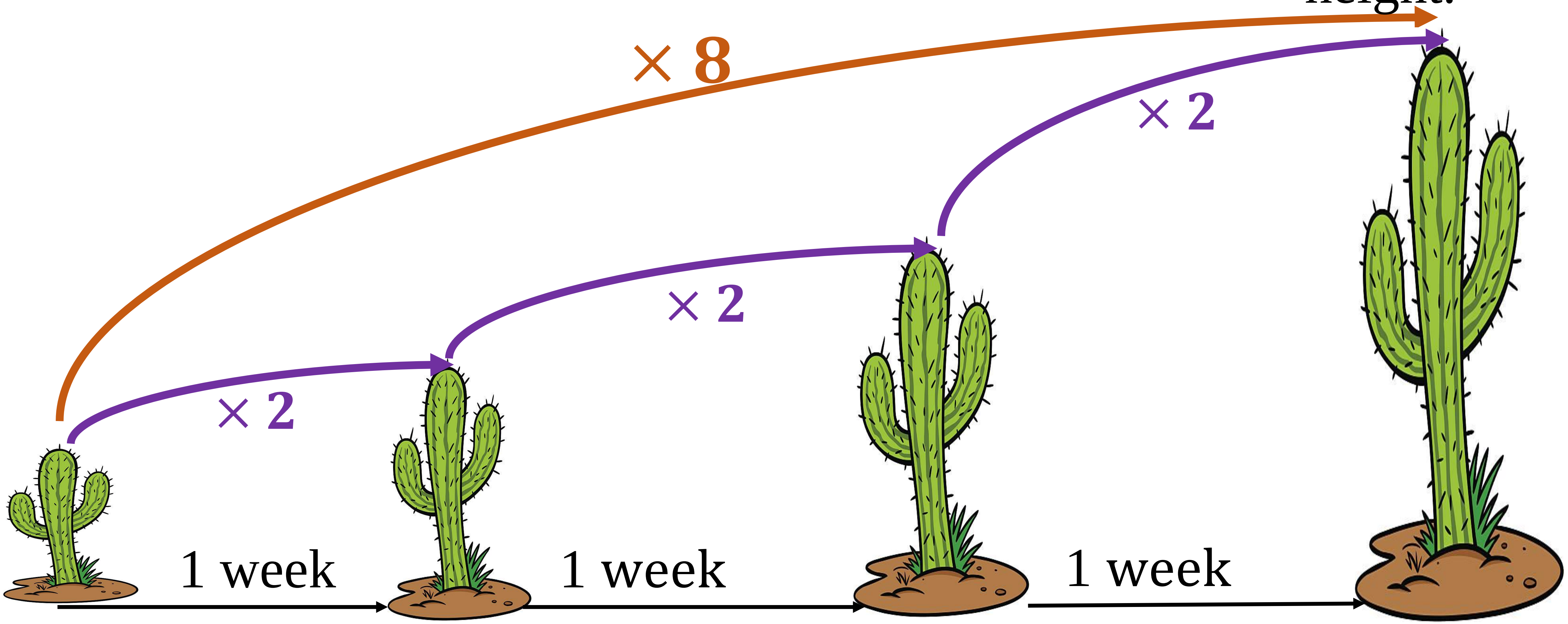
How many factors

$\overbrace{\text{How many factors of}}^{\log_2}$ the number **2** does the number **16** have?	**4**	$\log_2 16$
$\overbrace{\text{How many factors of}}^{\log_2}$ the number **2** does the number **32** have?	**5**	$\log_2 32$
$\overbrace{\text{How many factors of}}^{\log_2}$ the number **2** does the number **25** have?	?	$\log_2 25$

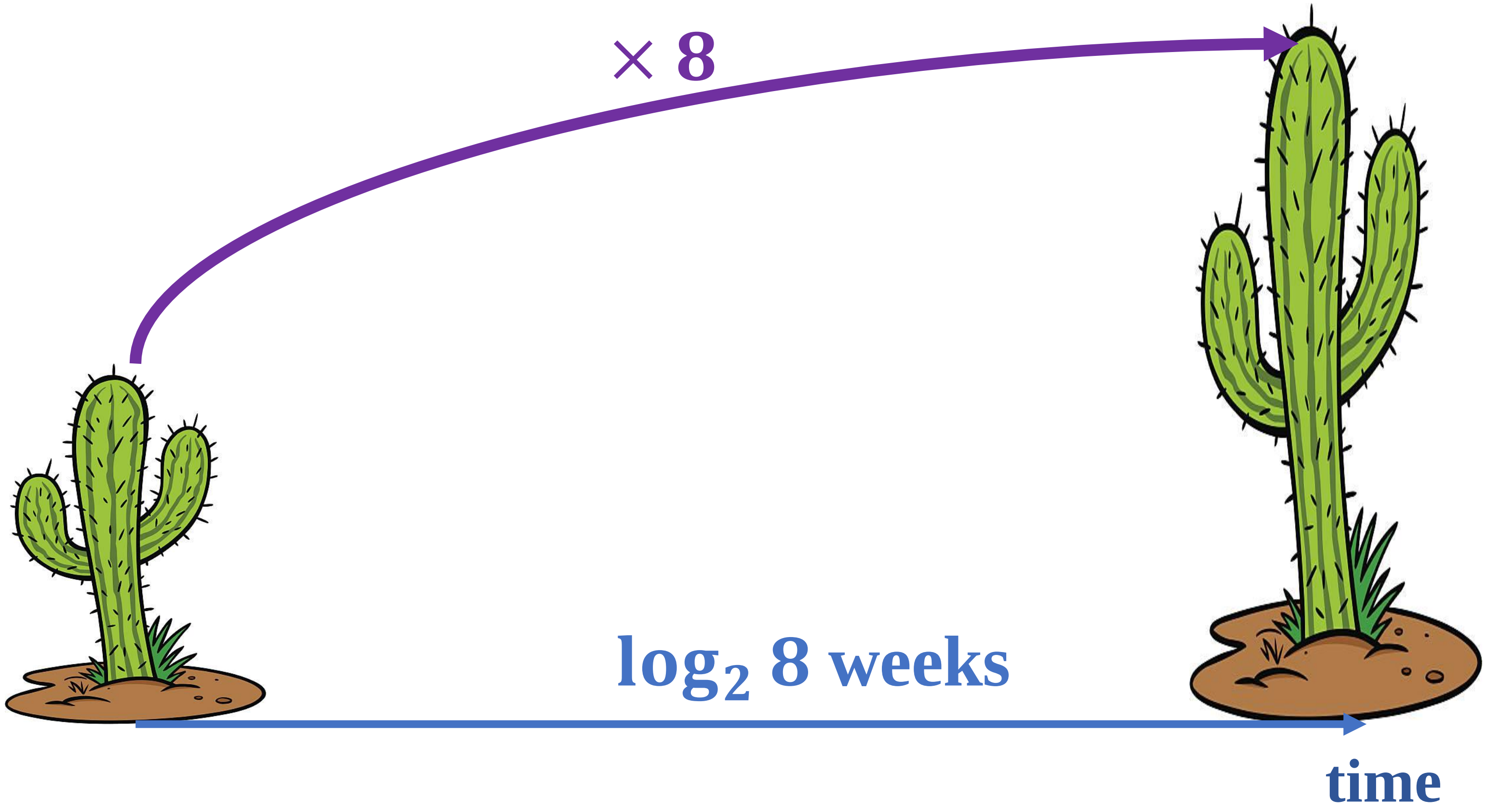
How can we describe $\log_2 8$ in the cactus story?

$\log_2 8$ is the number of factors of **2** that we have in the number **8**.

$\log_2 8$ is the number of weeks the cactus needs to reach **8 times** its height.

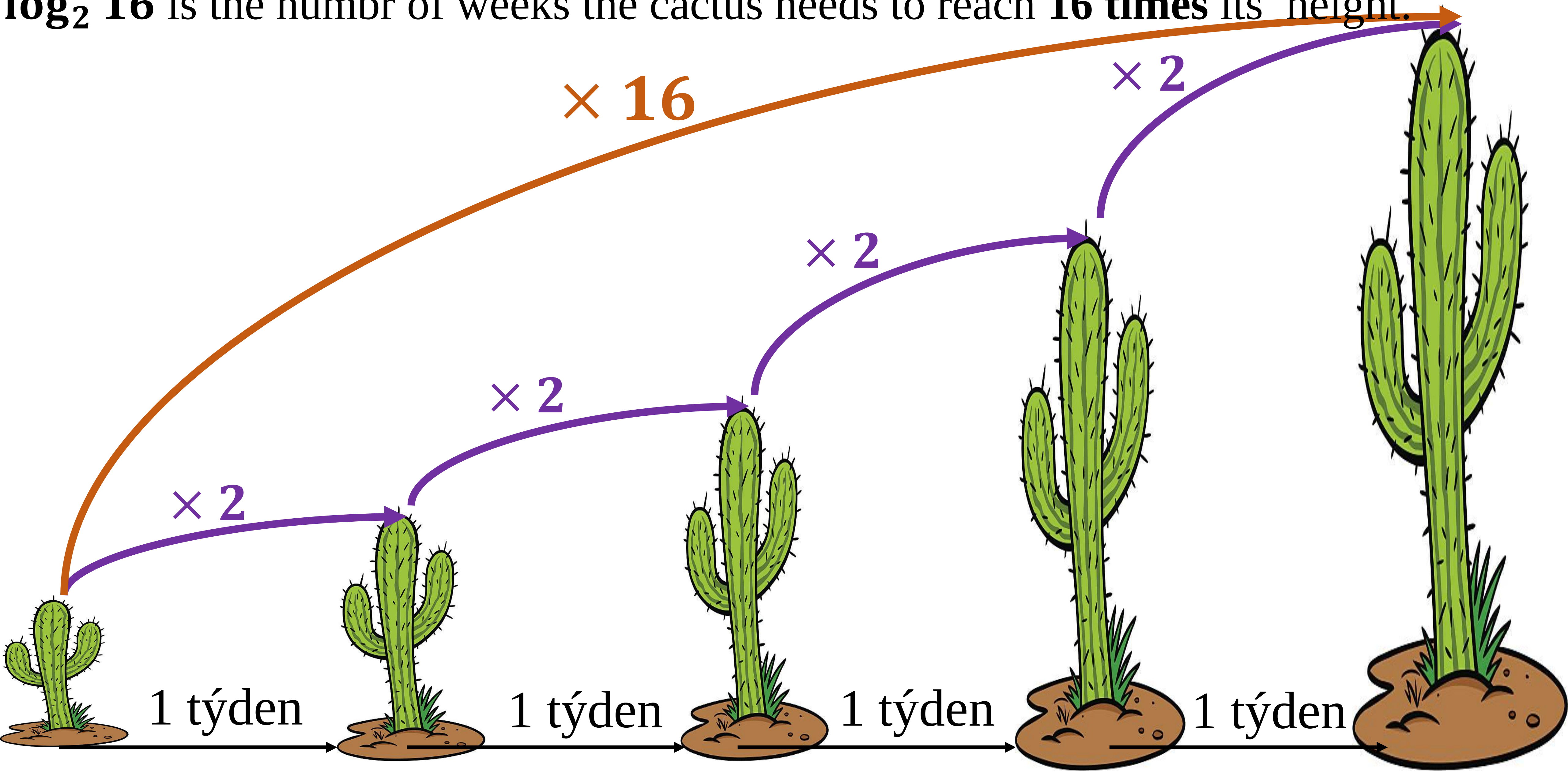


$\mathbf{\log_2 8}$ is **the number of weeks** the cactus needs to reach **8 times** its height.

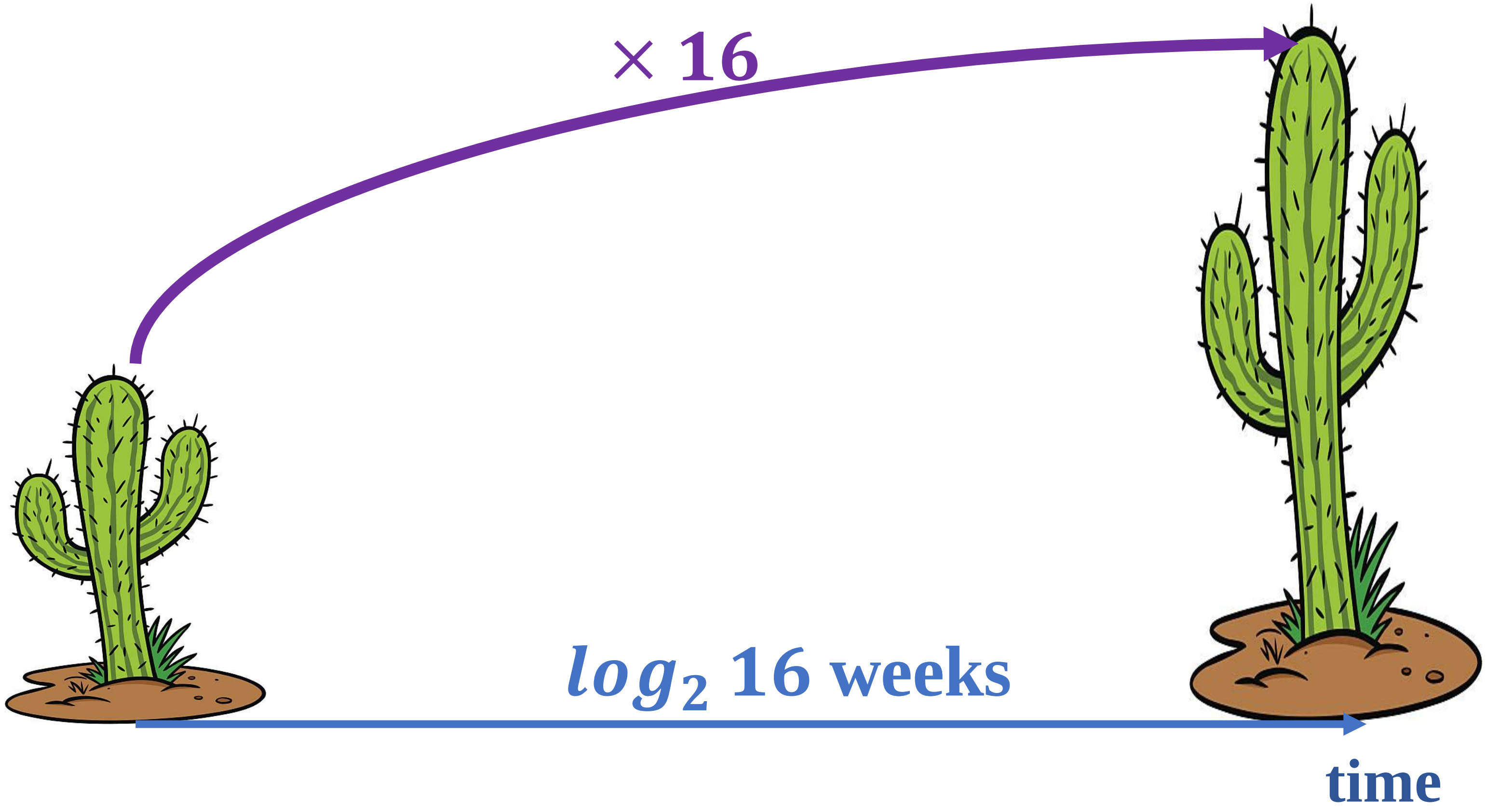


How can we describe $\mathbf{log_2 16}$ in the cactus story?

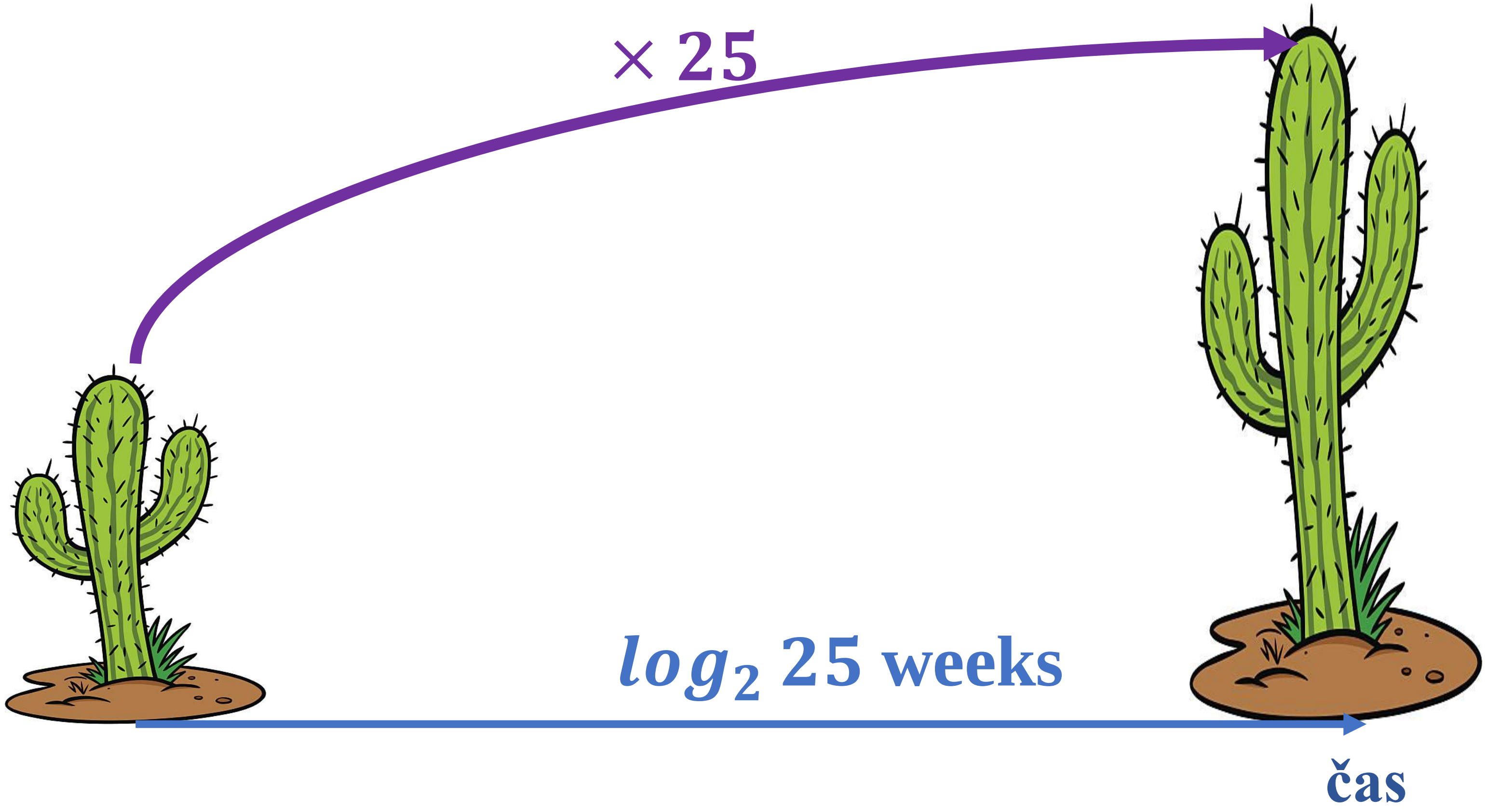
$\mathbf{log_2 16}$ is the numbr of weeks the cactus needs to reach **16 times** its height.



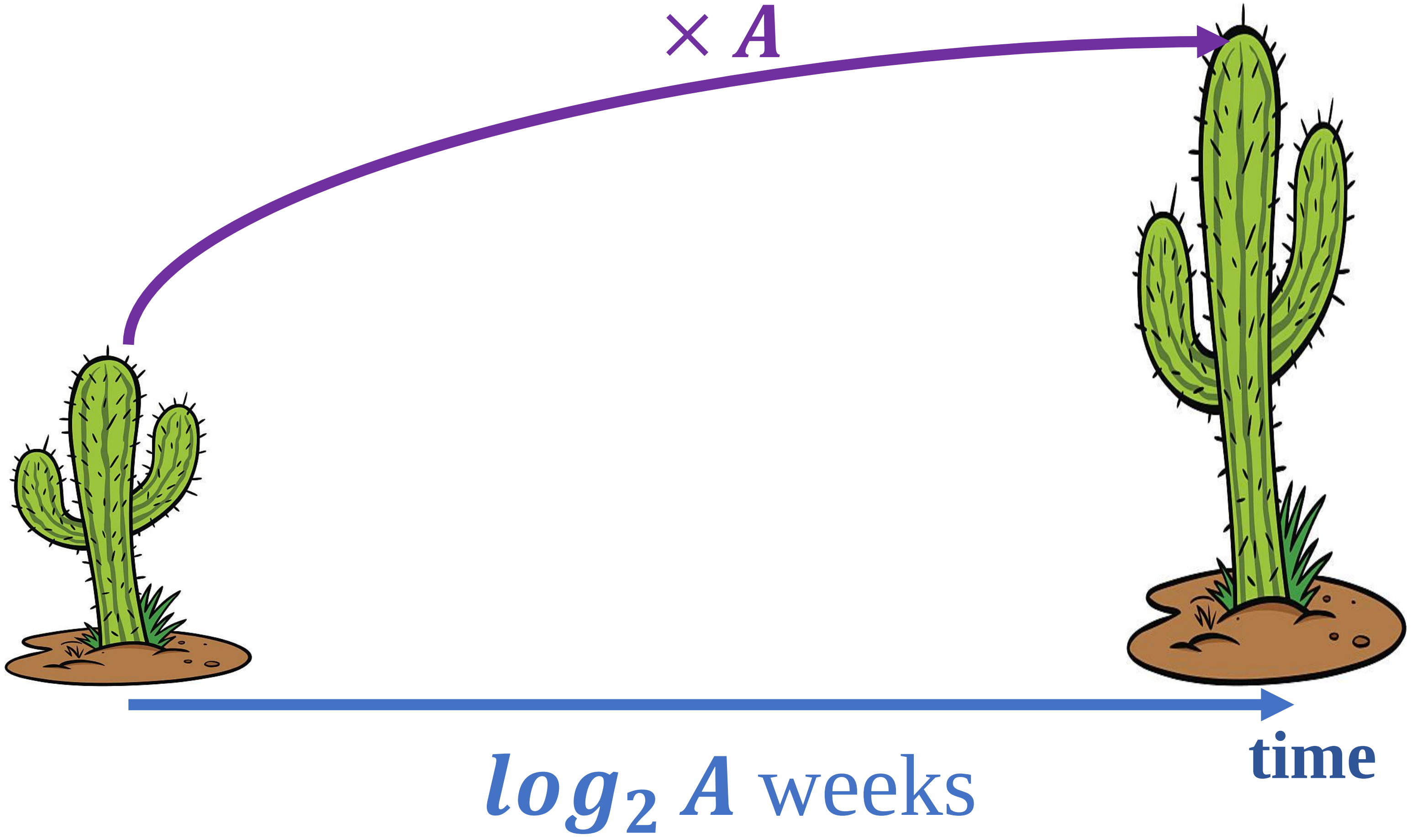
$\mathbf{log_2}$ **16** is **the number of weeks** the cactus needs to reach **16 times** its height.



$\log_2 25$ is **the number of weeks** the cactus needs to reach **25 times** its height.

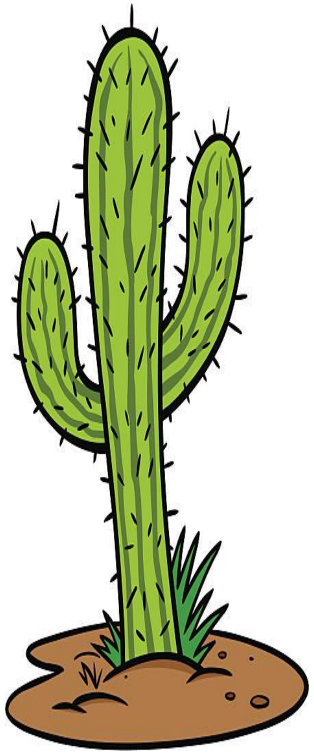


$\boldsymbol{log_2 A}$ expresses the number of weeks the cactus needs to reach $\boldsymbol{A}$**-times** its height.

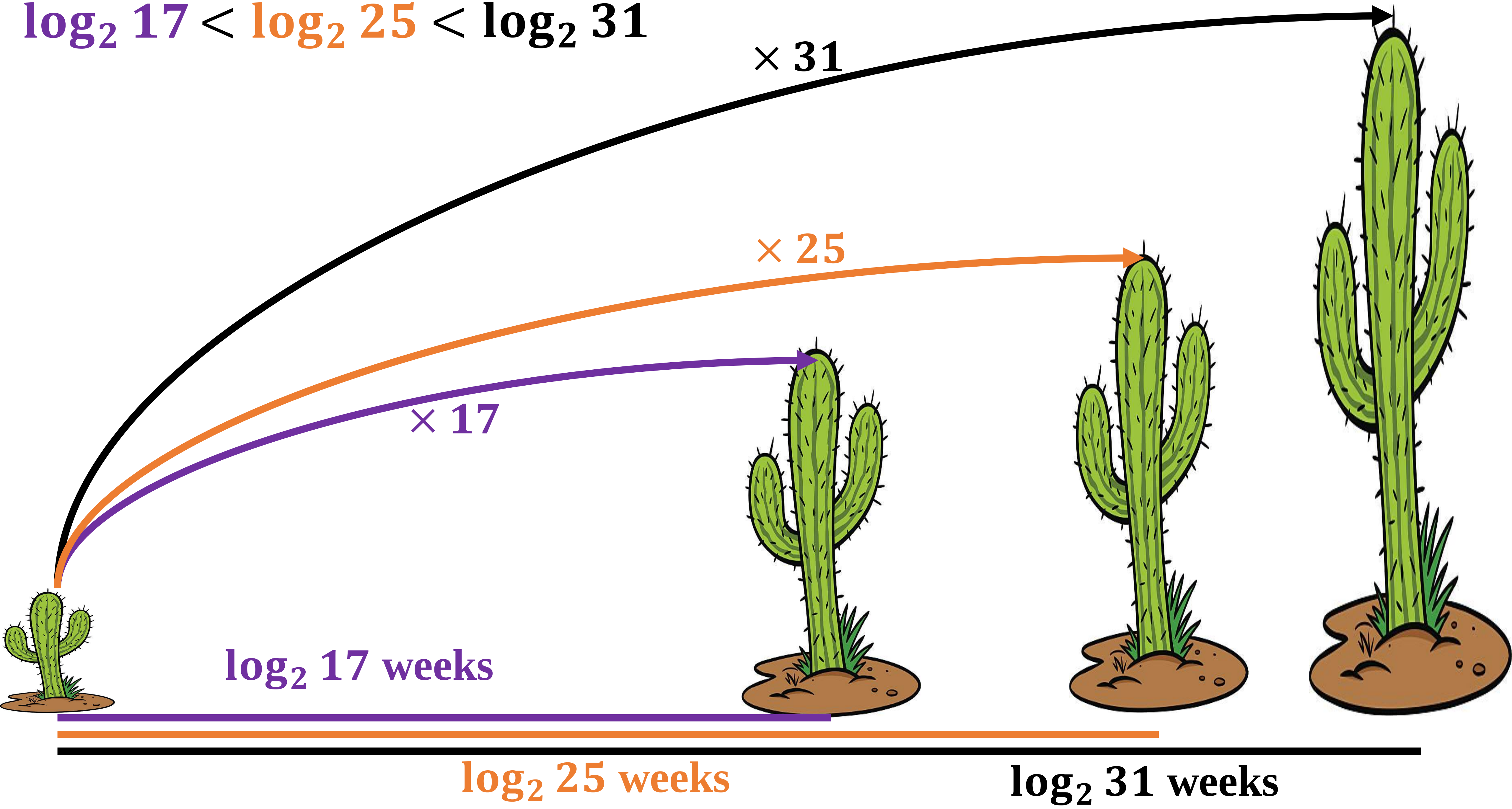


$$\log_2 17 < \log_2 25 < \log_2 31$$

$$4,0875 < 4,644 < 4,954$$

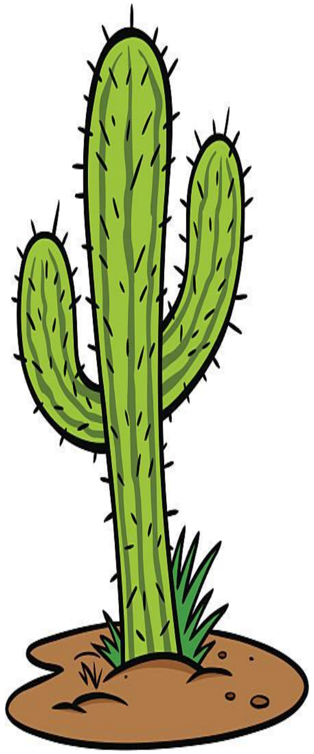


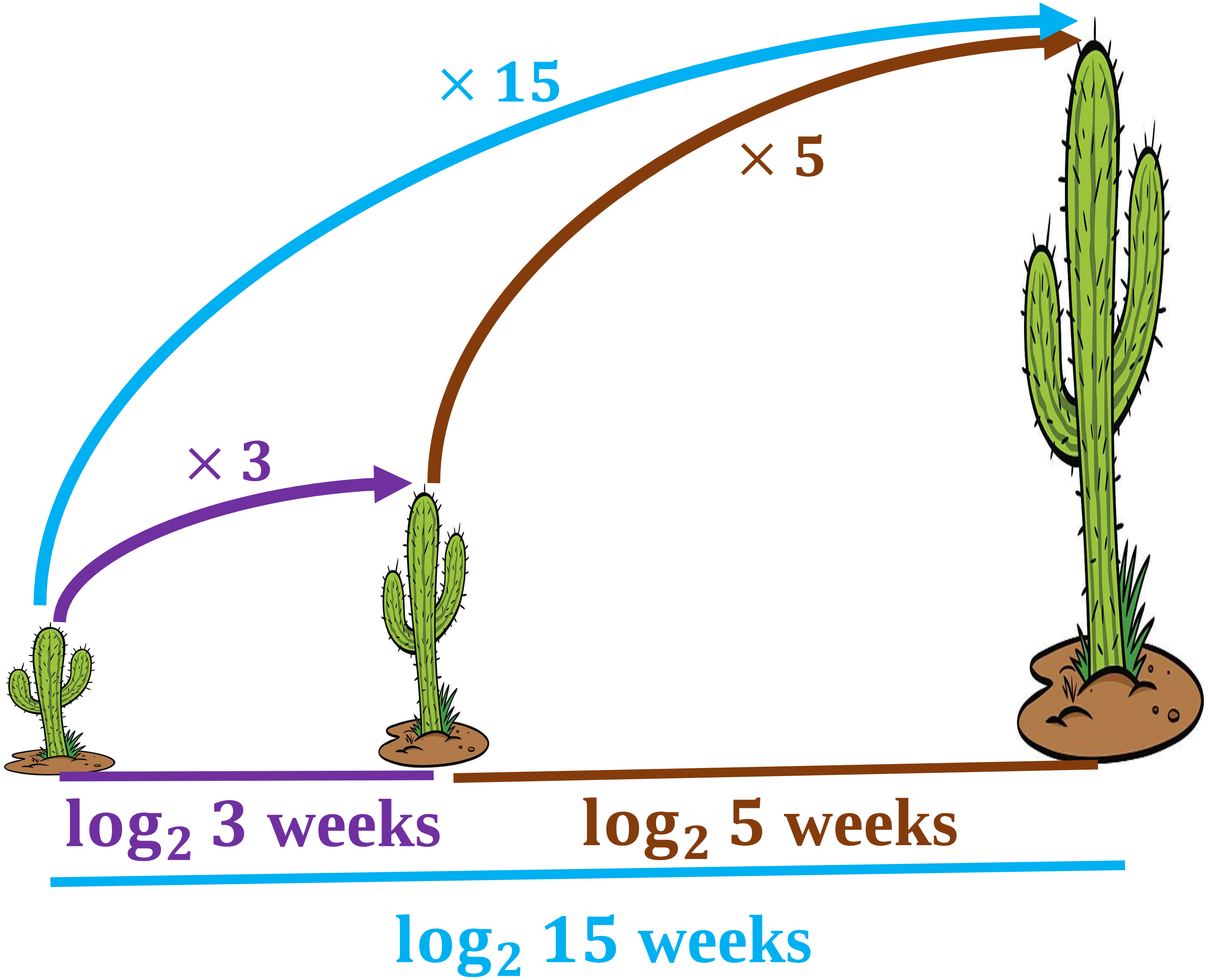
$$\log_2 17 < \log_2 25 < \log_2 31$$



$$\boxed{\log_b X + \log_b Y = \log_b XY}$$

$$\log_2 3 + \log_2 5 = \log_2 15$$





$\times\ 15$

$\times\ 5$

$\times\ 3$

$\log_2 3$ weeks

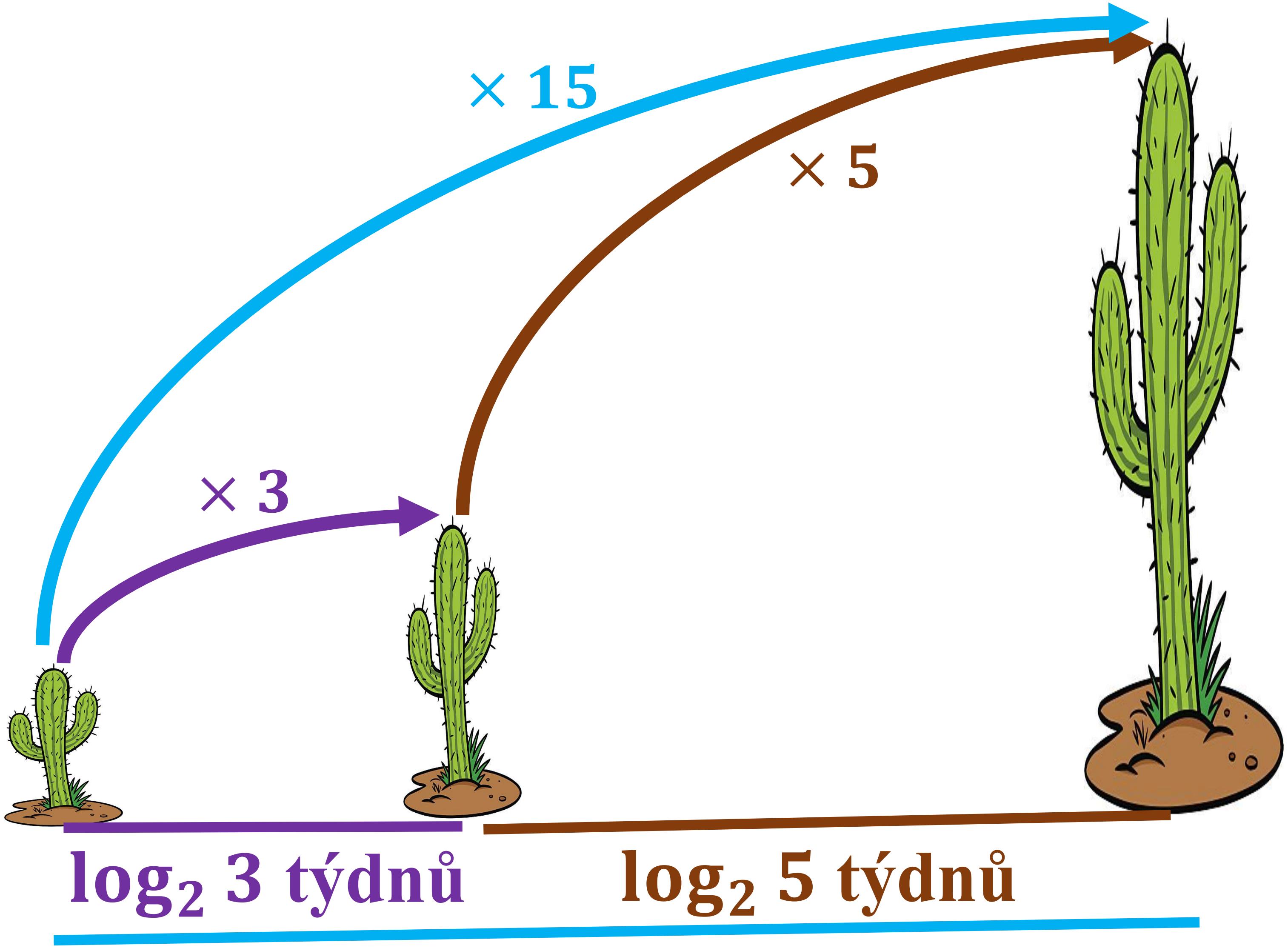
$\log_2 5$ weeks

$\log_2 15$ weeks

$$\log_2 3 + \log_2 5$$

$$=$$

$$\log_2 15$$



× 15

× 5

× 3

$\log_2 3$ týdnů

$\log_2 5$ týdnů

$\log_2 15$ týdnů

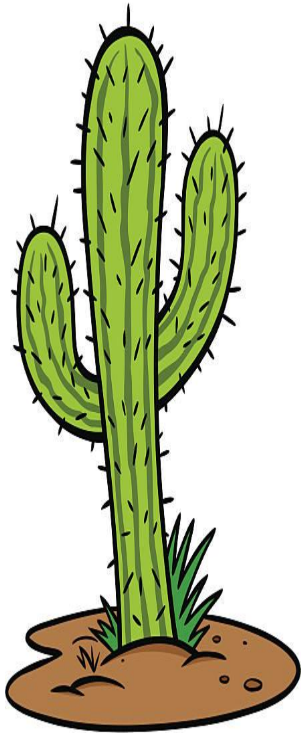
$$\log_2 3 + \log_2 5$$

$$=$$

$$\log_2 15$$

$$\boxed{\log_b X - \log_b Y = \log_b \frac{X}{Y}}$$

$$\log_2 12 - \log_2 4 = \log_2 3$$



References

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Thank you for your attention!

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