FUNCTIONAL ANALYSIS FOR PHYSICISTS: EXERCISE PROBLEMS

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Week 1.

PROBLEM 1.1. Let $A \in \mathbb{R}^{n\times n}$ be given. The following assertions are equivalent:

- (i) A is non-singular (the equation $Ax = b$ has one and only one solution for each $b \in \mathbb{R}^n$);
- (ii) the mapping $x \mapsto Ax$ is injective (the equation $Ax = b$ has at most one solution for each $b \in \mathbb{R}^n$);
- (iii) the mapping $x \mapsto Ax$ is surjective (the equation $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$).

In the following exercise we shall demonstrate that in the infinite-dimensional case (ii) and (iii) are not any more equivalent.

Consider mapping $T\colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ given by prescription

$$T\colon f(x) \mapsto f(x^2), \qquad x \in [0,1].$$

- (i) Verify that this is a correct definition and that the mapping T is linear.
- (ii) Show that $T - \mathrm{Id}$ is not injective.
- (iii) Show that $T + \mathrm{Id}$ is injective.
- (iv) Show that $T + \mathrm{Id}$ is not surjective.

Solution.

- (ii) As T is linear, it is sufficient to show that there is a non-trivial solution of the homogeneous equation $(T - \mathrm{Id})f = 0$. This is indeed the case, as any constant function, e.g., $f \equiv 1$, is a solution.
- (iii) Analogously, to show injectivity of $T + \mathrm{Id}$, we have to show that the only solution of the homogenous equation $(T + \mathrm{Id})f = 0$ is the zero function. Using the equation repeatedly, we obtain

  $$f(x) = -f(x^2) = f(x^4) = -f(x^8) = f(x^{16}) = \cdots .$$

  The first equality in particular implies that $f(0) = f(1) = 0$. By induction, for a fixed $a \in (0,1)$, we have, for any $n \in \mathbb{N}$, that

  $$f(a) = (-1)^n f(a^{2^n}) \to 0 \qquad \text{as } n \to \infty,$$

  with the limit due to continuity of f. This shows that $f(a) = 0$. As a was arbitrary from $(0,1)$, we conclude that $f \equiv 0$.
- (iv) To show that $T + \mathrm{Id}$ is not surjective, we need to show that there exists $g \in \mathcal{C}([0,1])$ such that the equation $(T + \mathrm{Id})f = g$ does not have a solution $f \in \mathcal{C}([0,1])$. Assume there is a solution. We have

  $$f(x^2) = g(x) - f(x), \qquad x \in [0,1],$$

  which yields, with change of variable,

  $$f(x) = g(x^{1/2}) - f(x^{1/2}), \qquad x \in [0,1],$$

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and, after recursive application of the equation,

$$f(x) = g(x^{1/2}) - g(x^{1/4}) + f(x^{1/4}), \qquad x \in [0,1],$$

$$\vdots$$

$$f(x) = \sum_{j=1}^{n} (-1)^{j-1} g(x^{2^{-j}}) + (-1)^n f(x^{2^{-n}}), \qquad x \in [0,1]. \tag{1.1}$$

Set $a \coloneqq 1/2$ and suppose that $g\colon [0,1] \to \mathbb{R}$ is a piecewise affine function interpolating the values

$$\begin{aligned} g(0) &\coloneqq 0, \\ g(a^{2^{-j}}) &\coloneqq \frac{(-1)^{j-1}}{j} \qquad \text{for } j \in \mathbb{N}, \\ g(1) &\coloneqq 0. \end{aligned}$$

It is left as a homework to show that $g \in \mathcal{C}([0,1])$. Substituting this choice of g into (1.1) yields, for $x \coloneqq a$,

$$f(a) = \sum_{j=1}^{n} \frac{1}{j} + (-1)^n f(a^{2^{-n}}).$$

The left-hand side is supposed to be a finite number by the required continuity of f, the first term on the right-hand side diverges as $n \to \infty$, and the last term goes to zero, which is the desired contradiction. ∎

Quiz 1.2. Let X and Y be nonempty sets and let a mapping $f\colon X \to Y$ be given. Which of the following possibilities are true?

(A) We say that f is injective if $f(x) = f(y)$ for some x, $y \in X$ implies that $x = y$.
(B) The equation $f(x) = b$ has at least one solution $x \in X$ for every $b \in Y$ if and only if f is surjective.
(C) Suppose that the equation $f(x) = b$ has a unique solution $x \in X$ for every $b \in Y$. Then f is surjective.
(D) Suppose that there exist $b \in Y$ such that the equation $f(x) = b$ does not have a solution. Then f is not surjective.
(E) Suppose that there exist $b \in Y$ such that $f(x) = b$ and $f(y) = b$ for some x, $y \in X$ with $x \neq y$. Then f is not injective.

Quiz 1.3. Let $A \in \mathbb{R}^{n\times n}$ be a real matrix and let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be the mapping given by $f(v) = Av$. Which of the following possibilities are true?

(A) If $\det A = 0$ then f is injective.
(B) f is surjective if and only f is injective.
(C) There exists $k \in \mathbb{N}$ such that the equation

$$(\underbrace{f \circ \ldots \circ f}_{k \text{ times}})(v) = 0$$

has a nonzero solution $v \in \mathbb{R}^n$.
(D) $\dim \ker f = \operatorname{rank} A^\top$.

QUIZ 1.4. Let $n \in \mathbb{N}$ be arbitrary and let $v,\, w \in \mathbb{C}^n$ be given vectors. Which of the following are true?

(A) Rank of $vw^\top$ is 1.
(B) Rank of $vw^\top$ is at most 1.
(C) Rank of $vw^\top$ is n.
(D) $\det(vw^\top) = 0$.
(E) $\det(vw^\top) = 1$.
(F) $\det(vw^\top) = w^\top v$.

PROBLEM 1.5.

(i) For a $p \geq 1$ consider the set of sequences

$$\ell_p := \bigl\{\{x_k\}_{k=1}^\infty \subset \mathbb{R},\ \textstyle\sum_{k>0} |x_k|^p < \infty\bigr\}.$$

What is the relation between ℓ_p and ℓ_q given $1 \leq p < q < \infty$?

(ii) Let $\Omega := (0,1)$. For a given $p \geq 1$ consider the set of p-integrable functions

$$L^p(\Omega) := \bigl\{u\colon \Omega \to \mathbb{R} \text{ measurable},\ \textstyle\int_\Omega |u|^p < \infty\bigr\}.$$

What is the relation between $L^p(\Omega)$ and $L^q(\Omega)$ given $1 \leq p < q < \infty$?

(iii) What is the relation between $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ given $1 \leq p < q < \infty$?

Solution.

(i) Let $\{y_k\}_{k=1}^\infty$ be arbitrary such that $\sum_k |y_k|^p = 1$. Then $|y_k| \leq 1$ for all $k \in \mathbb{N}$ and hence

$$\sum_{k\in\mathbb{N}} |y_k|^q \leq \sum_{k\in\mathbb{N}} |y_k|^p = 1. \tag{1.2}$$

Now for an arbitrary nonzero $x \in \ell_p$, set $y := \frac{x}{(\sum |x_k|^p)^{1/p}}$, which satisfies $\sum_k |y_k|^p = 1$, and hence (1.2) can be used for this y. After little rearrangement one gets $(\sum_k |x_k|^q)^{1/q} \leq (\sum_k |x_k|^p)^{1/p}$, which proves the inclusion $\ell_p \subset \ell_q$.

(ii) Hölder's inequality, for $r \geq 1$,

$$\int_\Omega |fg| \leq \Bigl(\int_\Omega |f|^r\Bigr)^{1/r}\Bigl(\int_\Omega |g|^s\Bigr)^{1/s}, \qquad \frac{1}{r} + \frac{1}{s} = 1,$$

gives for $f := |u|^p$, $g := 1$, and $r := q/p$

$$\int_\Omega |u|^p \leq \Bigl(\int_\Omega |u|^q\Bigr)^{p/q} |\Omega|^{1-p/q}.$$

After rearrangement,

$$\Bigl(\int_\Omega |u|^p\Bigr)^{1/p} \leq |\Omega|^{1/p-1/q}\Bigl(\int_\Omega |u|^q\Bigr)^{1/q},$$

which shows that $L^q(\Omega) \subset L^p(\Omega)$ whenever $|\Omega| < \infty$.

(iii) For $\Omega = \mathbb{R}$ the above argument does not work and clearly there are functions from $L^p(\mathbb{R})$ which are not in $L^q(\mathbb{R})$ and vice versa. For $u(x) := \Xi_{(0,1)} x^{-1/p+\varepsilon}$, where Ξ_M denotes the characteristic function of set $M \subset \mathbb{R}$, it is $L^p(\mathbb{R}) \ni u \notin L^q(\mathbb{R})$ if $\varepsilon > 0$ is chosen sufficiently small. On the other hand, for $v(x) := \Xi_{(1,\infty)} x^{-1/q-\varepsilon}$ with $\varepsilon > 0$ sufficiently small, it is $L^p(\mathbb{R}) \not\ni v \in L^q(\mathbb{R})$. □

Week 2.

Problem 2.1. Decide which of the following are normed spaces. If so, determine whether they are Banach.

(i) $(\mathbb{R}^3, \|\cdot\|_{1/2})$ for

$$\|x\|_{1/2} = \Big(\sum_{j=1}^{3} |x_j|^{1/2}\Big)^2.$$

(ii) $(\mathbb{R}, \|\cdot\|_t)$ for

$$\|x\|_t = \begin{cases} 3x & \text{if } x \geq 0, \\ -x & \text{otherwise.} \end{cases}$$

(iii) The space of polynomials of degree at most 2 with

$$\|p\| := |p(1)| + |p'(1)| + \tfrac{1}{2}|p''(1)|$$

(iv) The space of all polynomials with the maximum norm $\|p\|_\infty = \max\limits_{x\in[0,1]} |p(x)|$.

Solution.

(iv) The normed space $(\mathcal{P}, \|\cdot\|_\infty)$ of all polynomials on $[0,1]$ is not complete. The sequence of polynomials $\sum_{j=0}^{n} x^j/j!$, $n = 1, 2, \ldots$ converges uniformly in $[0,1]$, i.e., in the $\|\cdot\|_\infty$ norm, to $\exp(x) \notin \mathcal{P}$. □

Quiz 2.2. Let $f\colon [0,1] \to \mathbb{R}$ be Lebesgue integrable. Which of the following are true?

(A) $\int_0^x f(t)\,\mathrm{d}t$ is continuous on $[0,1]$.
(B) $\int_0^x f(t)\,\mathrm{d}t$ is uniformly continuous on $[0,1]$.
(C) $\int_0^x f(t)\,\mathrm{d}t$ is absolutely continuous on $[0,1]$.
(D) $\int_0^x f(t)\,\mathrm{d}t$ is Lipschitz continuous on $[0,1]$.
(E) $\int_0^x f(t)\,\mathrm{d}t$ is continuously differentiable on $[0,1]$.

Problem 2.3.

(i) Show that every subspace of a normed space is also a normed space (under the same norm).
(ii) Show that every closed subspace of a Banach space is also a Banach space (under the same norm).

Denote by ℓ_∞ the set of all bounded sequences of real or complex numbers, c the set of all convergent sequences of real or complex numbers, c_0 the set of all null (convergent to zero) sequences, and c_{00} the set of all eventually zero sequences (sequences with finitely many nonzero elements). Consider the supremum norm $\|x\|_\infty := \sup_{k>0} |x_k|$ and show that

(iii) $(\ell_\infty, \|\cdot\|_\infty)$ is a Banach space,
(iv) c is a closed subspace of $(\ell_\infty, \|\cdot\|_\infty)$,
(v) c_0 is a closed subspace of $(c, \|\cdot\|_\infty)$, and
(vi) c_{00} is a subspace of $(c_0, \|\cdot\|_\infty)$ which is not closed.

Solution.

(iii) We leave the task to verify that $(\ell_\infty, \|\cdot\|_\infty)$ is a normed space for the reader and proceed with completeness. Suppose that $\{x^n\}_{n=1}^\infty \subset \ell_\infty$ is a Cauchy sequence, i.e., for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|x^m - x^n\|_\infty < \varepsilon$ for all m, $n > N$, or equivalently, using the definition of $\|\cdot\|_\infty$,

$$|x_k^n - x_k^m| < \varepsilon \qquad \text{for all } m,\, n > N \text{ and all } k \in \mathbb{N}. \tag{2.1}$$

In particular, for a fixed $k \in \mathbb{N}$ the number sequence $\{x_k^n\}_{n=1}^\infty \subset \mathbb{R}$ is Cauchy and hence convergent to $x_k \coloneqq \lim_{n\to\infty} x_k^n$. Taking the limit $m \to \infty$ in (2.1) yields that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_k^n - x_k| \le \varepsilon \qquad \text{for all } n > N \text{ and all } k \in \mathbb{N}, \tag{2.2}$$

which can be rewritten as $\|x^n - x\|_\infty \to 0$ as $n \to \infty$ where $x \coloneqq \{x_k\}_{k=1}^\infty$. Let us finish by verifying that $x \in \ell_\infty$. Indeed, fixing $\epsilon > 0$ arbitrarily, (2.2) implies that for some $N \in \mathbb{N}$

$$\big||x_k| - |x_k^{N+1}|\big| \le \varepsilon \qquad \text{for all } k \in \mathbb{N},$$

and in turn $|x_k| \le |x_k^{N+1}| + \varepsilon$ for all $k \in \mathbb{N}$. As $x^{N+1} \in \ell^\infty$ and ε is fixed, one immediately gets that $x \in \ell_\infty$.

(iv) Let us show the closedeness. Suppose that $\{x^n\}_{n=1}^\infty \subset c$ is a convergent sequence (in the $\|\cdot\|_\infty$ norm), i.e., $\|x^n - x\|_\infty \to 0$ as $n \to \infty$ and $x \in \ell_\infty$ due to its completeness. We shall show that $x \in c$. Let us fix $\varepsilon > 0$ to an arbitrary value. By the uniform convergence $x^n \to x$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_k^n - x_k| < \frac{\varepsilon}{3} \qquad \text{for all } n \ge N_\varepsilon \text{ and all } k \in \mathbb{N}.$$

The number sequence $\{x_k^{N_\varepsilon}\}_{k=1}^\infty$ is convergent by the hypothesis $x^{N_\varepsilon} \in c$, i.e., (for the above chosen $\varepsilon > 0$) there exists $K \in \mathbb{N}$ such that

$$|x_k^{N_\varepsilon} - x_\ell^{N_\varepsilon}| < \frac{\varepsilon}{3} \qquad \text{for all } k,\, \ell > K.$$

Altogether, for arbitrary $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$|x_k - x_\ell| \le |x_k - x_k^{N_\varepsilon}| + |x_k^{N_\varepsilon} - x_\ell^{N_\varepsilon}| + |x_\ell^{N_\varepsilon} - x_\ell| < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon$$

for all k, $\ell > K$. In the other words, the number sequence $\{x_k\}_{k=1}^\infty$ is Cauchy and hence $x \in c$.

(v) Let us show the closedeness. Suppose that $\{x^n\}_{n=1}^\infty \subset c_0$ is a convergent sequence (in the $\|\cdot\|_\infty$ norm), i.e., $\|x^n - x\|_\infty \to 0$ as $n \to \infty$ and $x \in c$ as $(c, \|\cdot\|_\infty)$ is a Banach space by virtue of the previous task (iv). We shall show that $x \in c_0$. Let us fix $\varepsilon > 0$ to an arbitrary value. By the uniform convergence $x^n \to x$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_k^n - x_k| < \frac{\varepsilon}{2} \qquad \text{for all } n \ge N_\varepsilon \text{ and all } k \in \mathbb{N}.$$

The number sequence $\{x_k^{N_\varepsilon}\}_{k=1}^\infty$ is null (convergent to zero) by the hypothesis $x^{N_\varepsilon} \in c_0$, i.e., (for the above chosen $\varepsilon > 0$) there exists $K \in \mathbb{N}$ such that

$$|x_k^{N_\varepsilon}| < \frac{\varepsilon}{2} \qquad \text{for all } k > K.$$

Altogether, for arbitrary $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$|x_k| \le |x_k - x_k^{N_\varepsilon}| + |x_k^{N_\varepsilon}| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

for all $k > K$. In the other words, the number sequence $\{x_k\}_{k=1}^\infty$ is null and hence $x \in c_0$.

(vi) The sequence $\{(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots)\}_{n=1}^\infty \subset c_{00}$ converges in the supremum norm to $(1, \frac{1}{2}, \frac{1}{3}, \dots) \in c_0$, which is not an element of c_{00}. Hence c_{00} is not closed in $(c_0, \|\cdot\|_\infty)$. ☐

HOMEWORK 2.4.

(i) Show that, for a fixed $p \in [1, \infty)$, c_{00} is dense in the Banach space $(\ell_p, \|\cdot\|_p)$, where

$$\|x\|_p = \Big(\sum_{j=1}^\infty |x_i|^p\Big)^{\frac{1}{p}}.$$

(ii) Show that the closure of c_{00} in the supremum norm $\|\cdot\|_\infty$ coincides with c_0.

HOMEWORK 2.5. We say a subset V of a metric space is (sequentially) *compact* if every sequence in V has a convergent subsequence with the limit in V.

Let X be a Banach space, a set $A \subset X$ be closed, and a set $B \subset X$ be compact. Show that the set $A + B \coloneqq \{x + y,\, x \in A,\, y \in B\}$ is closed in X.

HOMEWORK 2.6. Let

$$f_n(x) \coloneqq \begin{cases} \frac{1}{n} & \text{if } x \in (0, n), \\ 0 & \text{otherwise.} \end{cases}$$

For every $p \in [1, \infty]$, determine whether $\{f_n\}$ has a limit in $(L^p(\mathbb{R}), \|\cdot\|_p)$,

$$\|f\|_p = \Big(\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x\Big)^{\frac{1}{p}}, \qquad p \in [1, \infty),$$

$$\|f\|_\infty = \operatorname{ess\,sup}_{\mathbb{R}} |f(x)|.$$

HOMEWORK 2.7. Consider X, the set of continuous functions $u \colon [0, \infty) \to \mathbb{R}$ such that

$$\|u\|_e \coloneqq \sup_{x \in [0,\infty)} e^x |u(x)|$$

is finite. Show that $(X, \|\cdot\|_e)$ is a normed space and determine whether it is complete.

HOMEWORK 2.8. For each $n \in \mathbb{N}$, let the sequence $\{x_k^n\}_{k=1}^\infty \subset \mathbb{R}$ be given by

$$x_k^n = \frac{k+1}{k^2+2} + \frac{n+1}{n^2 k}, \qquad k \in \mathbb{N}.$$

(i) Determine whether x^n, $n \in \mathbb{N}$, belong to c_0, ℓ_1, ℓ_2, ℓ_3, and ℓ_∞.

(ii) Determine whether the sequence $\{x^n\}_{n=1}^\infty$ converges in Banach spaces $(c_0, \|\cdot\|_\infty)$ and $(\ell_\infty, \|\cdot\|_\infty)$. If yes, establish the limit.

Week 3.

PROBLEM 3.1.

(i) Consider $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$, the vector space of continuous functions on $[0,1]$ equipped with the maximum norm $\|u\|_\infty := \max_{x\in[0,1]} |u(x)|$. Think through that this is a normed space. Show that it is complete.

(ii) Show that $(\mathcal{C}([0,1]), \|\cdot\|_1)$, $\|u\|_1 := \int_0^1 |u(x)|\,\mathrm{d}x$ is a normed space which is not complete. As a counterexample consider the sequence

$$f_n(x) := \begin{cases} 0, & x \le \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}(x - \frac{1}{2}) + \frac{1}{2}, & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n}, \\ 1, & \frac{1}{2} + \frac{1}{n} \le x. \end{cases}$$

(iii) THEOREM 3.2 (Arzelà–Ascoli). *Let a sequence of continuous functions* $\{f_n\}_{n=1}^\infty \subset \mathcal{C}([0,1])$ *be given.*

If $\{f_n\}_{n=1}^\infty$ *is* uniformly bounded, *i.e., there exists* $M > 0$ *such that*

$$\|f_n\|_\infty \le M,$$

and uniformly equicontinuous, *i.e., for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $x, y \in [0,1]$ *with* $|x - y| < \delta$ *it holds*

$$\sup_{n\in\mathbb{N}} |f_n(x) - f_n(y)| \le \varepsilon,$$

then there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ *that converges uniformly on* $[0,1]$.

The converse is true as well in the following sense: If every subsequence of $\{f_n\}_{n=1}^\infty$ *admits a uniformly convergent subsequence then* $\{f_n\}_{n=1}^\infty$ *is uniformly bounded and uniformly equicontinuous.*

Use the theorem to judge whether $\{f_n\}_{n=1}^\infty$ from (ii) is uniformly convergent.

Solution.

(i) This is the *uniform limit theorem*. Its proof uses the $\varepsilon/3$ strategy as in Problem 2.3 (iv).

(ii) The pointwise limit

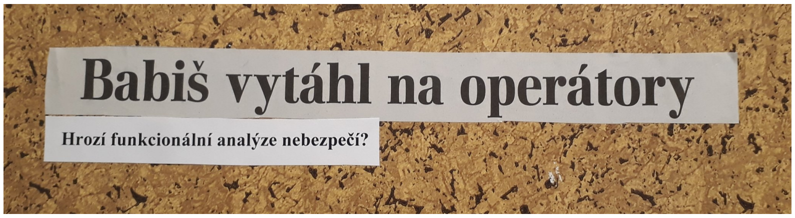
$$f(x) := \begin{cases} 0, & x < \frac{1}{2}, \\ 1, & x > \frac{1}{2}, \end{cases}$$

does not belong to $\mathcal{C}([0,1])$, but a straightforward computation shows that $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

(iii) Clearly it is $\|f_n\| \le 1$ for all $n \in \mathbb{N}$, so the sequence is uniformly bounded. On the other hand, the modulus of continuity blows up with $n \to \infty$: For arbitrary $\varepsilon > 0$, it is

$$|f_n(x) - f_n(y)| \le \varepsilon \qquad \text{if } |x - y| < \frac{2\varepsilon}{n}.$$

So $\delta > 0$ cannot be chosen independent of $n \in \mathbb{N}$, and the uniform equicontinuity is violated. Hence, according to the theorem, it cannot be that $\|f_n - f\|_\infty \to 0$. (The reader should think through that selecting a subsequence and/or assuming a different limit instead of f is of no help here.) □



Babiš vytáhl na operátory

Hrozí funkcionální analýze nebezpečí?

Problem 3.3.

(i) Let $A \in \mathbb{R}^{n\times m}$ be a given matrix. Consider the mapping $T_A\colon \mathbb{R}^m \to \mathbb{R}^n\colon x \mapsto Ax$. Verify that T_A is a linear bounded operator w.r.t. the Euclidean norm on $\mathbb{R}^m$ and $\mathbb{R}^n$. Does the operator norm $\|T_A\|$ coincide with some matrix norm of A? Is the norm attained for some $x \in \mathbb{R}^m$?

(ii) **(Diagonal operator on ℓ_p).** Let an arbitrary sequence $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ and $p \in [1,\infty]$ be given. Consider the operator $T : \ell_p \to \ell_p$ given by

$$T(x_1, x_2, x_3, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \ldots).$$

Equip ℓ_p with its usual norm $\|x\|_p := \left(\sum_{k=1}^{\infty} |x|^p\right)^{1/p}$. Compute the norm of $T : (\ell_p, \|\cdot\|_p) \to (\ell_p, \|\cdot\|_p)$. When is the operator bounded?

(iii) For real functions on $[0,1]$, consider the differentiation mapping $f \mapsto f'$. This is clearly a linear operator. Consider the sequence $\{f_n\}_{n=1}^{\infty}$, $f_n(x) = \sin(nx)$. Compute $\|f_n\|_\infty$ and $\|f_n'\|_\infty$. Is the operator $(\mathcal{C}^1([0,1]), \|\cdot\|_\infty) \to (\mathcal{C}([0,1]), \|\cdot\|_\infty)\colon f \mapsto f'$ bounded?

(iv) **(Shift operator on L^p).** Let $a \in \mathbb{R}$ and $p \in [1,\infty]$ be given. Consider the mapping T_a given for a $f \in L^p(\mathbb{R})$ by prescription

$$(T_a f)(x) = f(x-a) \qquad \text{for a.e. } x \in \mathbb{R}.$$

Clearly T_a is a linear operator and $\|T_a f\|_p = \|f\|_p$. Hence, $T_a\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded with $\|T_a\| = 1$. Observe that T_a is a bijection.

(v) **(Shift operators on ℓ_p).** For any $1 \le p \le \infty$, define the *right shift* $S_R\colon \ell_p \to \ell_p$ and the *left shift* $S_L\colon \ell_p \to \ell_p$ by

$$S_R(x_1, x_2, x_3, \ldots) := (0, x_1, x_2, \ldots),$$
$$S_L(x_1, x_2, x_3, \ldots) := (x_2, x_3, x_4, \ldots).$$

Verify that these are bounded linear operators, compute their norms, and check whether they are injective or surjective.

(vi) **(Multiplication operator).** Let $\Omega \subset \mathbb{R}$ be open and let $g \in L^\infty(\Omega)$ be given. Consider the *multiplication operator*, which, for an $f \in L^p(\Omega)$, $1 \le p \le \infty$, is given by

$$(M_g f)(x) = f(x)\, g(x) \qquad \text{for a.e. } x \in \mathbb{R}.$$

Compute the norm of $M_g\colon L^p(\Omega) \to L^p(\Omega)$.

(vii) Consider the indefinite integral operator, for $f \in \mathcal{C}([a,b])$, $a < b$, given by

$$Tf(x) = \int_a^x f(s)\,\mathrm{d}s \qquad \text{for all } x \in [a,b].$$

Show that $T\colon (\mathcal{C}([a,b]), \|\cdot\|_\infty) \to (\mathcal{C}([a,b]), \|\cdot\|_\infty)$ is bounded and that $\|T\| = b - a$.

Do you know how can be the range of $T\colon L^1((a,b)) \to \mathcal{C}([a,b])$ described?

Solution.

(i) We have

$$\|T_A\| = \sup_{\|x\|_2 \le 1} \|T_A(x)\|_2 = \sup_{\|x\|_2 \le 1} \|Ax\|_2 = \|A\|_2,$$

the spectral norm of A. The norm is attained by any dominant right singular vector: If $A = \sum_i \sigma_i u_i v_i^\top$ with $\sigma_1 \ge \sigma_2 \ge \cdots \ge 0$ and $\{u_i\}_i$, $\{v_i\}_i$ orthonormal systems, then $\|Av_1\| = \|\sigma_1 v_1\| = \sigma_1 = \|A\|_2$.

(ii) Suppose that $p < \infty$. We estimate

$$\|Tx\|_p^p = \sum_{i=1}^{\infty} |\lambda_i x_i|^p \le \sup_{i\in\mathbb{N}} |\lambda_i|^p \sum_{i=1}^{\infty} |x_i|^p = \|\lambda\|_\infty^p \|x\|_p^p,$$

which implies that

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|_p}{\|x\|_p} \le \|\lambda\|_\infty.$$

For

$$x^n = (0,\dots,0,\underset{\substack{\uparrow \\ n\text{-th position}}}{1},0,\dots),$$

it is

$$\|x^n\|_p = 1 \qquad \text{and} \qquad \|Tx^n\|_p = |\lambda_n|.$$

Hence

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|_p}{\|x\|_p} \ge \sup_{n\in\mathbb{N}} \frac{\|Tx^n\|_p}{\|x^n\|_p} = \sup_{n\in\mathbb{N}} |\lambda_n| = \|\lambda\|_\infty.$$

Using both inequalities we conclude that $\|T\| = \|\lambda\|_\infty$ and clearly T is bounded if and only if $\lambda \in \ell_\infty$. We leave the modifications necessary to handle the case $p = \infty$ for the reader.

(iii) It is

$$\|f_n\|_\infty = 1 \qquad \text{and} \qquad \|f_n'\|_\infty = n$$

and hence

$$\| \bullet' \| = \sup_{\|f\|_\infty \le 1} \|f'\|_\infty \ge \sup_{n\in\mathbb{N}} \|f_n'\|_\infty = \sup_{n\in\mathbb{N}} n = \infty,$$

that is the operator $f \mapsto f'$ is not bounded.

(v) Assume first $p < \infty$. We have

$$\|S_R x\|_p^p = \sum_{i=1}^{\infty} |x_i|^p = \|x\|_p^p,$$

$$\|S_L x\|_p^p = \sum_{i=2}^{\infty} |x_i|^p \le \|x\|_p^p.$$

The inequality becomes an equality if $x = (0, x_2, x_3, \dots)$. This together shows that $\|S_R\| = 1$ and $\|S_L\| = 1$. A minor modification shows the same for $p = \infty$.

Given any $p \in [1, \infty]$, the equation

$$S_R x = (1, 0, 0, \dots)$$

does not have a solution $x \in \ell_p$ and hence S_R is not surjective. On the other hand the equation

$$S_R x = 0$$

only has a trivial solution $x = 0$ and hence S_R is injective.

Given arbitrary $p \in [1, \infty]$ and $y = (y_1, y_2, \dots) \in \ell_p$, the equation

$$S_L x = y$$

has a solution, for example, $x = (0, y_1, y_2, y_3, \dots)$ and hence S_L is surjective. On the other hand, the equation

$$S_L x = 0$$

has a non-trivial solution $x = (1, 0, 0, \dots)$ and hence S_L is not injective. ∎

HOMEWORK 3.4. Consider the vector space of real sequences $\mathbb{R}^\infty$. Show that the function $d \colon \mathbb{R}^\infty \times \mathbb{R}^\infty \to [0, +\infty]$ given by

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}, \qquad x = \{x_j\}_{j=1}^{\infty},\ y = \{y_j\}_{j=1}^{\infty},$$

is a metric (distance) on $\mathbb{R}^\infty$. Apply the following proposition to $f \colon s \mapsto \frac{s}{1+s}$.

LEMMA. *Let $f \colon [0, \infty) \to \mathbb{R}$ be concave such that $f(0) \geq 0$. Then $f(a + b) \leq f(a) + f(b)$ for all a, $b \geq 0$.*

Proof. By hypotheses, we have, with $t \in [0, \infty)$ and $0 \leq \lambda \leq 1$, that

$$f(\lambda t) = f\big(\lambda t + (1 - \lambda) 0\big) \geq \lambda f(t) + (1 - \lambda) f(0) \geq \lambda f(t).$$

Hence,

$$\begin{aligned} f(a) + f(b) &= f\Big(\frac{a}{a+b}(a+b)\Big) + f\Big(\frac{b}{a+b}(a+b)\Big) \\ &\geq \frac{a}{a+b} f(a+b) + \frac{b}{a+b} f(a+b) = f(a+b). \end{aligned}$$

∎

Solution to Homework 3.4. First we verify that $d(x, y) < +\infty$ for all x, $y \in \mathbb{R}^\infty$. The inequality $\frac{s}{1+s} \leq 1$ for all $s \geq 0$ implies that

$$d(x, y) \leq \sum_{j=1}^{\infty} 2^{-j} = 1.$$

The symmetry $d(x, y) = d(y, x)$ is clear and also clearly $d(x, x) = 0$. On the other hand if $d(x, y) = 0$ then $|x_j - y_j| = 0$ for all $j \in \mathbb{N}$ and hence $x = y$. It remains to verify the triangle inequality.

The function $s \mapsto \frac{s}{1+s}$ is increasing, concave, and non-negative on $[0, +\infty)$. So by the monotonocity and the lemma, for arbitrary a, b, and $c \geq 0$ such that $c \leq a + b$, it is

$$\frac{c}{1 + c} \leq \frac{a + b}{1 + a + b} \leq \frac{a}{1 + a} + \frac{b}{1 + b}.$$

Applying the inequality to $a = |x_j - y_j|$, $b = |y_j - z_j|$, and $c = |x_j - z_j|$ we obtain

$$\begin{aligned} d(x,z) &= \sum_{j=1}^{\infty} 2^{-j} \frac{|x_j - z_j|}{1 + |x_j - z_j|} \\ &\leq \sum_{j=1}^{\infty} 2^{-j} \left(\frac{|x_j - y_j|}{1 + |x_j - y_j|} + \frac{|y_j - z_j|}{1 + |y_j - z_j|} \right) = d(x,y) + d(y,z). \quad \square \end{aligned}$$

Homework 3.5 (discrete metric space). Let X be a set and define, for x, $y \in X$,

$$d(x,y) := \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Show that $\big(X, d(\cdot,\cdot)\big)$ is a metric space. Describe all open balls and spheres in the space.

Week 4.

Quiz 4.1. For $p \geq 1$ let ℓ_p denote the set of all sequences $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}$ such that

$$\sum_{j=1}^{\infty} |x_j|^p < \infty.$$

For $p \geq 1$ and an open set $\Omega \subset \mathbb{R}$, let $L^p(\Omega)$ denote the set of all measurable functions $f\colon \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} |f(x)|^p \,\mathrm{d}x < \infty.$$

Which of the following possibilities are true?

(A) $\ell_1 \supset \ell_2$;
(B) $L^1\big((0,1)\big) \subset L^2\big((0,1)\big)$;
(C) $L^1\big((0,+\infty)\big) \subset L^2\big((0,+\infty)\big)$;
(D) $L^1\big((0,+\infty)\big) \supset L^2\big((0,+\infty)\big)$.

Quiz 4.2. For arbitrary $a \in \mathbb{R}$, let $c(a)$ denote the set of all convergent sequences of real numbers with the limit equal to a. Let d be the set of all divergent sequences of real numbers. Which of the following possibilities are true?

(A) $c(0)$ is a vector space;
(B) $c(1)$ is a vector space;
(C) d is a vector space.

Quiz 4.3. Consider the sequence of functions $f_n\colon (0,1) \to \mathbb{R}$, $n = 1, 2, \ldots$, given by

$$f_n(x) = x^{-1+\frac{1}{n}}.$$

For each $x \in (0,1)$, denote

$$f(x) := \lim_{n\to\infty} f_n(x) \qquad \text{if it exists, otherwise} \qquad f(x) := \infty.$$

Which of the following possibilities are true?

(A) For every $x \in (0,1)$, the number sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges;
(B) $f_n \in L^1\big((0,1)\big)$ for every $n \in \mathbb{N}$;
(C) $f \in L^1\big((0,1)\big)$;
(D) the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is bounded in $L^1\big((0,1)\big)$;
(E) $\{f_n\}_{n=1}^{\infty}$ contains a subsequence which converges in $L^1\big((0,1)\big)$.

Problem 4.4. On the Banach space $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$ consider the following operators and decide whether they are linear and bounded:

(i) $Tf(x) = f(\cos^2(x))$,
(ii) $Tf(x) = \cos^2(f(x))$,
(iii) $Tf(x) = f(0)f'(x)$,
(iv) $Tf(x) = (x-1)xf(0) + \int_0^x f(s)\,\mathrm{d}s$,
(v) $Tf(x) = y(x)$, where y is the solution of the initial value problem $y' + y = f$ in $(0,1)$, $y(0) = 0$.

Solution.

(i) T is clearly linear and also bounded. Indeed, for arbitrary $x \in [0,1]$, it is

$$|f(\cos^2 x)| \le \max_{t\in[0,1]} |f(t)| = \|f\|_\infty.$$

Hence $\|Tf\|_\infty = \max_{x\in[0,1]} |f(\cos^2 x)| \le \|f\|_\infty$, which shows that $\|T\| \le 1$. Choosing $f \equiv 1$ shows that $\|T\| = 1$.

(ii) T is clearly non-linear.

(iii) T is clearly non-linear.

(iv) T is linear and, for arbitrary $x \in [0,1]$,

$$\begin{aligned}|Tf(x)| \le |f(0)|\,|x-1|\,|x| + \left|\int_0^x f(s)\,\mathrm{d}s\right| &\le \tfrac{1}{4}|f(0)| + \int_0^1 |f(s)|\,\mathrm{d}s \\ &\le \tfrac{1}{4}\|f\|_\infty + \|f\|_\infty.\end{aligned}$$

Hence $\|T\| \le \frac{5}{4}$ and T is bounded.

(v) For f_1, $f_2 \in \mathcal{C}([0,1])$, consider y_1, $y_2 \in \mathcal{C}([0,1])$ such that

$$\begin{aligned} y_1' + y_1 &= f_1 \quad \text{in } (0,1), & y_1(0) &= 0,\\ y_2' + y_2 &= f_2 \quad \text{in } (0,1), & y_2(0) &= 0.\end{aligned}$$

Due to the linearity of the equations, we have

$$(y_1+y_2)' + (y_1+y_2) = (f_1+f_2) \qquad \text{in } (0,1), \qquad (y_1+y_2)(0) = 0,$$

which shows that $T(f_1+f_2) = Tf_1 + Tf_2$. Proceeding similarly for homogeneity, we get that T is linear.

It is readily verified that T has the explicit representation

$$Tf(x) = \int_0^x \exp(t-x)\,f(t)\,\mathrm{d}t.$$

Hence, for any $x \in [0,1]$,

$$|Tf(x)| \le \int_0^x \exp(t-x)\,|f(t)|\,\mathrm{d}t \le \int_0^x |f(t)|\,\mathrm{d}t \le \|f\|_\infty.$$

Hence, T is bounded with $\|T\| \le 1$. ◻

DEFINITION. *A function $p\colon X \to \mathbb{R}$ on a vector space X over $\mathbb{R}$ (or $\mathbb{C}$) is called a* seminorm *if it satisties the following properties:*

(i) $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and $x \in X$;

(ii) $p(x+y) \le p(x) + p(y)$ for all x, $y \in X$.

A sequence of seminorms $\{p_j\}_{j=1}^{\infty}$ is called separating *if for every nonzero $x \in X$ there exists $j \in \mathbb{N}$ such that $p_j(x) > 0$.*

Observe that (ii) implies that $p(0) = 0$ and (i) and (ii) together imply that $p(x) \ge 0$ for all $x \in X$. Indeed, $0 = p(0) = p\big(x + (-x)\big) \le p(x) + p(-x) = p(x) + p(x) = 2p(x)$.

PROBLEM 4.5. Adapt the solution of Homework 3.4 to show the following proposition.

PROPOSITION (metric generated by seminorms). *Let $\{p_j\}_{j=1}^{\infty}$ be a separating sequence of seminorms on a vector space X. Then*

$$d(x,y) = \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(x-y)}{1 + p_j(x-y)} \tag{4.1}$$

is a metric (distance) on X.

DEFINITION. *Let X be a vector space over either $\mathbb{R}$ or $\mathbb{C}$ and $\{p_j\}_{j=1}^{\infty}$ be a separating sequence of seminorms on X. If the metric space $\big(X, d(\cdot,\cdot)\big)$ with d given by (4.1) is complete, then $\big(X, \{p_j\}_{j=1}^{\infty}\big)$ is called a* Fréchet space.

EXAMPLE 4.6 (examples of Fréchet spaces [1, Examples 2.25, 2.26]).

Week 5.

EXAMPLE 5.1 (Schwartz space of rapidly decreasing functions [2]). The Schwartz space (the space of rapidly decreasing functions)

$$\mathcal{S}(\mathbb{R}^n) \coloneqq \big\{u \in C^{\infty}(\mathbb{R}^n),\ \|x^{\beta}\partial_{\alpha} u\|_{\infty} < \infty \text{ for all multiindices } \alpha,\ \beta\big\}$$

is a Fréchet space (without proof) when equipped with the sequence of seminorms $\{p_j\}_{j=0}^{\infty}$,

$$p_j(u) \coloneqq \sum_{|\alpha|,|\beta| \le j} \|x^{\beta}\partial_{\alpha} u\|_{\infty},$$

or, for example, $\{q_j\}_{j=0}^{\infty}$,

$$q_j(u) \coloneqq \max_{|\alpha| \le j} \|(1+|x|^2)^j \partial_{\alpha} u\|_{\infty}.$$

These two generate the same topology. Significance of the space is that (i) Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is one-to-one, (ii) Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n)' \to \mathcal{S}(\mathbb{R}^n)'$ on tempered distributions $\mathcal{S}(\mathbb{R}^n)'$ is naturally defined (by moving $\mathcal{F}$ to test functions), and (iii) as $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, $\mathcal{F}$ can be extended to $\hat{\mathcal{F}}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which is unitary. For details see [2].

PROBLEM 5.2 (Minkowski functional). Let X be a real normed space and $B \subset X$ be a nonempty convex open set containing the origin. Let the functional $p : X \to [0,\infty)$ be defined by

$$p(x) \coloneqq \inf\{\lambda > 0,\ x \in \lambda B\}, \qquad \text{for every } x \in X.$$

Show that

(i) there exists $M > 0$ such that $p(x) \le M\|x\|$ for all $x \in X$;
(ii) $B = \{x \in X,\, p(x) < 1\}$;
(iii) p is sublinear, i.e.,

$$\begin{aligned} p(\alpha x) &= \alpha p(x) && \text{for all } x \in X \text{ and } \alpha \ge 0 \text{ and} \\ p(x+y) &\le p(x) + p(y) && \text{for all } x,\, y \in X. \end{aligned}$$

Solution.

(i) By the hypothesis, there exists a ball $B_r := \{x \in X,\, \|x\| < r\}$ with certain $r > 0$ such that $B_r \subset B$. Hence

$$p(x) = \inf\{\lambda > 0,\, \tfrac{x}{\lambda} \in B\} \le \inf\{\lambda > 0,\, \tfrac{x}{\lambda} \in B_r\} = \frac{\|x\|}{r}.$$

(ii) To show "$\subset$", suppose that $x \in B$. As B is open, $(1+\delta)x \in B$ for some $\delta > 0$ small enough. In the other words, $\frac{x}{\lambda} \in B$ for $\lambda = \frac{1}{1+\delta}$, and hence

$$p(x) = \inf\{\lambda > 0,\, \tfrac{x}{\lambda} \in B\} \le \inf\Big\{\frac{1}{1+\delta}\Big\} = \frac{1}{1+\delta} < 1.$$

For the opposite inclusion, suppose that $p(x) < 1$. By the definition of p, there exists $0 < \beta < 1$ such that $\beta \in \{\lambda > 0,\, x/\lambda \in B\}$, and hence $x/\beta \in B$. As B is convex and contains the origin, we have

$$x = \beta\,\frac{x}{\beta} + (1-\beta)\,0 \in B.$$

(iii) We leave the task to verify positive homogeneity, $p(\alpha x) = \alpha p(x)$, for all $x \in X$ and $\alpha \ge 0$, up to the reader, so it remains to prove the triangle inequality. Suppose that $x,\, y \in X$ and fix $\varepsilon > 0$. Then for $\frac{x}{p(x)+\varepsilon}$, we have

$$p\left(\frac{x}{p(x)+\varepsilon}\right) = \frac{p(x)}{p(x)+\varepsilon} < 1,$$

where the equality follows from the positive homogeneity, and hence, by virtue of (ii), $\frac{x}{p(x)+\varepsilon} \in B$. Similarly, $\frac{y}{p(y)+\varepsilon} \in B$. By the convexity of B, it follows that, with arbitrary $0 < \mu < 1$,

$$\mu\,\frac{x}{p(x)+\varepsilon} + (1-\mu)\,\frac{y}{p(y)+\varepsilon} \in B.$$

Chossing $\mu := \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$ and using (ii) and the positive homogeneity yields

$$1 > p\left(\frac{x+y}{p(x)+p(y)+2\varepsilon}\right) = \frac{p(x+y)}{p(x)+p(y)+2\varepsilon}.$$

As ε was arbitrary, it is $p(x+y) \le p(x) + p(y)$. ☐

HOMEWORK 5.3 (epigraph). For a function $f\colon X \to \mathbb{R}$, its *epigraph* is defined as

$$\operatorname{epi} f := \{(x,y) \in X \times \mathbb{R},\, y \ge f(x)\}.$$

Show the following propositions.

LEMMA 5.4. *Let X be a convex subset of a real vector space and suppose that $f\colon X \to \mathbb{R}$ is convex. Then $\operatorname{epi} f$ is convex.*

If X is a normed space, the product $X\times\mathbb{R}$ is a normed space with, e.g., $\|(x,y)\|_{X\times\mathbb{R}} \coloneqq \|x\|_X + |y|$. Recall we say that a function $f\colon X\to\mathbb{R}$ is (norm) lower semicontinuous if $x_n\to x$ (in norm) implies $\liminf_{n\to\infty} f(x_n)\geq f(x)$.

Lemma 5.5. *Suppose that X is a normed space and $f\colon X\to\mathbb{R}$ is (norm) lower semicontinuous. Then $\operatorname{epi} f$ is (norm) closed.*

Week 6.

Theorem 6.1 (Hahn–Banach separation theorem). *Let A, B be nonempty disjoint convex subsets of a real normed space X.*

(i) If A is open then there exists $F\in X^*$ and $c\in\mathbb{R}$ such that

$$F(x) < c \leq F(y) \qquad \text{for all } x\in A \text{ and } y\in B.$$

If both A and B are open, then both of the inequalities can be taken as strict.

(ii) If A is closed and B is compact the there exists $F\in X^*$ and c_1, $c_2\in\mathbb{R}$ such that

$$F(x)\leq c_1 < c_2 \leq F(y) \qquad \text{for all } x\in A \text{ and } y\in B.$$

First we show part (i) for B singleton, i.e., we show the following claim.

Lemma 6.2. *If C is a nonempty convex open subset of a real normed space X and $z\in X\setminus C$, then there exists $F\in X^*$ with $F(x)<F(z)$ for all $x\in C$.*

Proof. For now assume that $0\in C$. Set $Z\coloneqq \operatorname{span}\{z\}$, which is one-dimensional as z is nonzero due to $C\cap\{z\}=\emptyset$ and $0\in C$. Define linear functional $F_0\colon Z\to\mathbb{R}$ by $F_0(tz)=t$ for $t\in\mathbb{R}$. We will show that F_0 is dominated on Z by the Minkowski functional $p_C\colon X\to[0,\infty)$ given by

$$p_C(x)=\inf\{\lambda>0,\, x\in\lambda C\} \qquad \text{for } x\in X.$$

It has been shown in Problem 5.2 that p_C is a norm-bounded sublinear function and that $C=\{x\in X,\, p_C(x)<1\}$. In particular it is $p_C(z)\geq 1$ as $z\notin C$. Hence

$$\begin{aligned} F_0(tz) &= t \leq t p_C(z) = p_C(tz) && \text{for all } t\geq 0,\\ F_0(tz) &= t\leq 0\leq p_C(tz) && \text{for all } t\leq 0, \end{aligned}$$

which shows that $F_0\leq p_C$ on Z. The Hahn–Banach theorem thus ensures that there exists linear $F\colon X\to\mathbb{R}$ such that

$$F=F_0 \text{ on } Z \qquad \text{and} \qquad F\leq p_C \text{ on } X.$$

Thus,

$$-M\|x\| \leq -p_C(-x)\leq F(x)\leq p_C(x)\leq M\|x\| \qquad \text{for any } x\in X,$$

where $M>0$ is the constant from the estimate $p_C(x)\leq M\|x\|$. Thus F is bounded. We also have for $x\in C$

$$F(x)\leq p_C(x) < 1 = F_0(z) = F(z),$$

which finishes the case $0\in C$.

If $0 \notin C$, choose any $x_0 \in C$ and set

$$\tilde{C} \coloneqq \{x - x_0 \in X,\, x \in C\} \qquad \text{and} \qquad \tilde{z} = z - x_0.$$

It is $0 \in \tilde{C}$ and $\tilde{z} \notin C$, hence, by virtue of the already proved case "$0 \in C$", there exists $F \in X^*$ with $F(\tilde{x}) < F(\tilde{z})$ for all $\tilde{x} \in \tilde{C}$. Adding $F(x_0)$ to both sides yields $F(x) = F(\tilde{x} + x_0) < F(\tilde{z} + x_0) = F(z)$ for all $x \in C$. ∎

Lemma 6.3. *Let X be a normed space and suppose that F is a nonzero linear functional defined on X. Then F is surjective. If M is an open subset of X, then $F(M)$ is open.*

Proof. As $F \neq 0$, there is $y \in X$ such that $F(y) = 1$. One-dimensional subspace $\operatorname{span}\{y\}$ is mapped by F onto $\mathbb{R}$ (or $\mathbb{C}$ if X is complex), thus F is surjective. Fix $x \in M$. We need to show that $\mathcal{B}_\varepsilon(F(x)) \subset F(M)$ for some $\varepsilon > 0$, where $\mathcal{B}_r(s)$ is an open ball of radius r around s in $\mathbb{R}$ (or $\mathbb{C}$). The set $\{x + ty,\, t \in \mathcal{B}_\varepsilon(0)\}$ is mapped by F onto $\mathcal{B}_\varepsilon(F(x))$ as

$$F(x + ty) = F(x) + tF(y) = F(x) + t.$$

Altogether we have, with notation $\mathcal{B}_r^X(z) = \{v \in X,\, \|v - z\| < r\}$,

$$\mathcal{B}_\varepsilon(F(x)) = F(\{x + ty,\, t \in \mathcal{B}_\varepsilon(0)\} \subset F(\mathcal{B}_\varepsilon^X(x)) \subset F(M),$$

where the last inclusion is true for small enough $\varepsilon > 0$ as M is open. ∎

Proof of Theorem 6.1. For A open, the set

$$C \coloneqq \bigcup_{y \in B} \{x - y \in X,\, x \in A\}$$

is open (as a union of open sets) and nonempty. Convexity of A and B easily imply that C is convex, which we leave for the reader to check. It is also $0 \notin C$ since A, B are disjoint. Lemma 6.2 provides $F \in X^*$ such that $F(\tilde{x}) < F(0) = 0$ for all $\tilde{x} \in C$, or equivalently $F(x) < F(y)$ for all $x \in A$ and $y \in B$. This implies that

$$F(x) \leq \sup F(A) \leq c \leq \inf F(B) \leq F(y) \qquad \text{for all } x \in A \text{ and } y \in B$$

holds true for some $c \in \mathbb{R}$. Lemma 6.3 shows that the left-most inequality is strict owing to A being open and analogously for the right-most inequality if B is additionally open.

It remains to prove part (ii). For $r > 0$ let

$$A_r \coloneqq \bigcup_{x \in A} \mathcal{B}_r(x), \qquad B_r \coloneqq \bigcup_{y \in B} \mathcal{B}_r(y),$$

where $\mathcal{B}_r(z) = \{v \in X,\, \|v - z\| < r\}$ is the open ball in X of radius r centered at z. We leave it as an exercise to verify that A_r, B_r are nonempty, convex, and open given any $r > 0$. We also claim that there exists $R > 0$ such that $A_r \cap B_r = \emptyset$ whenever $0 < r \leq R$. To see it, assume for contradiction that there are sequences $\{x_n\} \subset A$, $\{y_n\} \subset B$, $\{v_n\}$, $\{w_n\} \subset X$ such that

$$x_n + v_n = y_n + w_n \quad \text{for all } n \in \mathbb{N}, \qquad v_n \to 0, \qquad w_n \to 0.$$

Compactness of B implies that there is a subsequence of $\{y_n\}$ with $\{y_{n_k}\} \to y \in B$. Then

$$x_{n_k} = y_{n_k} + w_{n_k} - v_{n_k} \to y + 0 - 0 = y.$$

This shows that $\{x_{n_k}\} \subset A$ converges to y, which thus must be in A owing to its closedness. Altogether we have shown that y is an element of A and B at the same time, which is a contradiction with $A \cap B = \emptyset$. Thus the claim that A_R, B_R are disjoint for $R > 0$ small enough is thus proved. By part (i) A_R, B_R are separated by a plane, i.e., for some nonzero $F \in X^*$ and $c \in \mathbb{R}$, it is

$$F(x+v) < c < F(y+w) \qquad \text{for all } x \in A,\ y \in B,\ v,\ w \in X \text{ with } \|v\|,\ \|w\| < R.$$

This implies that, for any $x \in A$, $y \in B$,

$$F(x) + R\|F\| = \sup_{\|v\|<R} F(x+v) \le c \le \inf_{\|w\|<R} F(y+w) = F(y) - R\|F\|.$$

The result follows with $c_{1,2} := c \mp R\|F\|$ as $R\|F\| > 0$. □

Week 7.

PROBLEM 7.1. We say that a subset $M \subset X$ of a normed space X is *(sequentially) weakly closed* if every weakly convergent sequence $\{x_n\}_{n\ge 1} \subset M$ satisfies $x_n \rightharpoonup x \in M$. We can immediately see that a weakly closed set is closed. Indeed, suppose that $\{x_n\} \subset M$ conveges in norm to $x \in X$. Then $\{x_n\}$ converges weakly to the same x. As M is weakly closed, it is necessarily $x \in M$. The converse holds true for convex sets:

THEOREM 7.2. *A subset of a normed space that is closed and convex is weakly closed.*

We say that $f\colon X \to \mathbb{R}$ is *weakly lower semicontinuous* if the weak convergence $x_n \rightharpoonup x$ implies $\liminf_{n\to\infty} f(x_n) \ge f(x)$.

THEOREM 7.3. *Let f be a real-valued functional on a normed space which is lower semicontinuous and convex. Suppose additionally that f is bounded from below. Then f is weakly lower semicontinuous.*

COROLLARY 7.4. *Let V be a normed space. Then the norm $\|\cdot\|\colon V \to \mathbb{R}\colon x \mapsto \|x\|$ is weakly lower semicontinuous, i.e.,*

$$\liminf_{n\to\infty} \|x_n\| \ge \|x\| \qquad \textit{whenever } x_n \rightharpoonup x.$$

Prove the theorems and the corollary. Theorem 7.2 can be proved by contradiction, invoking the strict Hahn–Banach (strict) separation theorem (Theorem 6.1 (ii)). Theorem 7.3 follows from Lemmas 5.4 and 5.5 and Theorem 7.2.

Solution. We shall only show proofs for the case that the underlying normed space is a *real* normed space.

Proof of Theorem 7.2. We suppose, according to the hypotheses, that X is a real normed space and M is its closed convex subset. Suppose that $\{x_j\}_{j=1}^{\infty} \subset M$ converges weakly to $x \in X$. For the proof, we need to show that $x \in M$. For contradiction assume that $x \in X \setminus M$. Then by the Hahn–Banach separation theorem, Theorem 6.1 (ii), M and $\{x\}$ are strictly separated by a hyperplane. Indeed, the singleton set $\{x\}$ is convex and compact. So there is $F \in X^*$ such that $\sup_M F < F(x)$. But by virtue of the weak convergence it is $F(x_j) \to F(x)$ as $j \to \infty$. So

$$F(x) > \sup_M F \geq \sup_{j\in\mathbb{N}} F(x_j) \geq \lim_{j\to\infty} F(x_j) = F(x),$$

which is a contradiction. ☐

Proof of Theorem 7.3. We suppose, according to the hypotheses, that X is a real normed and $f\colon X \to \mathbb{R}$ is convex, lower semicontinuous, and bounded from below, i.e., $\inf_X f > -\infty$. According to Lemmas 5.4 and 5.5, the epigraph of f is a closed convex subset of the product space $X \times \mathbb{R}$. Thus, by Theorem 7.2, the epigraph is weakly closed.

Let $\{x_j\}_{j=1}^\infty \subset X$ be an arbitrary sequence that converges weakly in X towards some $x \in X$. Denote $L \coloneqq \liminf f(x_j)$, an element of the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$. For proof, we need to show that $f(x) \leq L$. If $L = +\infty$, the claim is true and the proof is finished. The case $L = -\infty$ would contradict the hypothesis of boundedness from below. Indeed,

$$-\infty < \inf_X f \leq \liminf_{j\to\infty} f(x_j) = L = -\infty.$$

Hence $L = -\infty$ cannot happen and it remains to deal with the case $L \in \mathbb{R}$.

Consider the sequence $\{x_j, f(x_j)\}_{j=1}^\infty$, which is a subset of the graph of f, which is in turn a subset of the epigraph. By the definition of limit inferior, there is a subsequence $f(x_{j_k}) \to L$ as $k \to \infty$. So in the product space $X \times \mathbb{R}$, the sequence $\{(x_{j_k}, f(x_{j_k}))\}_{k=1}^\infty$, a subset of the epigraph, goes weakly to (x, L). By the weak closedeness of the epigraph, the limit (x, L) also belongs to the epigraph, or equivalently, $L \geq f(x)$. ☐

Corollary 7.4 follows immediately from Theorem 7.3 once we verify its assumptions. From triangle inequality we get $\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\|$ when $0 \leq \lambda \leq 1$ and x, $y \in X$, and also $\big|\|x_n\| - \|x\|\big| \leq \|x_n - x\|$, which shows that $x \mapsto \|x\|$ is continuous (w.r.t. convergence in norm), in particular, lower semicontinuous. Norm is also bounded from below, $\|x\| \geq 0$. ☐

Problem 7.5 (complex Hahn–Banach theorem).

(i) Let V be a vector space over $\mathbb{C}$. Show that V is a vector space over $\mathbb{R}$.

(ii) Let $f\colon V \to \mathbb{C}$ be a linear functional on the complex vector space V. Define f_1, $f_2\colon V \to \mathbb{R}$ by

$$f_1(x) \coloneqq \operatorname{Re} f(x),$$
$$f_2(x) \coloneqq \operatorname{Im} f(x).$$

Show that f_1 and f_2 are linear functionals on V over $\mathbb{R}$, but they are not, in general, linear functionals on V over $\mathbb{C}$.

(iii) Show that $f_2(x) = -f_1(ix)$, and hence $f(x) = f_1(x) - if_1(ix)$.

(iv) Let X be a complex vector space, $p\colon X \to \mathbb{R}$ be a seminorm, and let $V \subset X$ be a subspace of X. Suppose that $f\colon V \to \mathbb{C}$ is linear such that $|f(x)| \leq p(x)$ on V. Apply the real version of Hahn–Banach theorem to construct a linear $F_1\colon X \to \mathbb{R}$, an extension of $f_1\colon V \to \mathbb{R}$, such that $|F_1| \leq p$ on X.

(v) From F_1 construct a linear $F\colon X \to \mathbb{C}$, an extension of $f\colon V \to \mathbb{C}$, and show that $|F| \leq p$ on X.

Solution.

(ii) For arbitrary $x, y \in V$, we have $f_1(x+y) = \operatorname{Re} f(x+y) = \operatorname{Re} f(x) + \operatorname{Re} f(y) = f_1(x) + f_1(y)$. As of homogeneity, we have $f_1(\lambda x) = \operatorname{Re} f(\lambda x) = \operatorname{Re}(\lambda f(x))$ for any $\lambda \in \mathbb{C}$. If λ is real, then the last expression equals $\lambda f_1(x)$, which shows that f_1 is linear on V over $\mathbb{R}$. On the other hand, homogeneity $f_1(\lambda x) = \lambda f_1(x)$ is clearly violated if, for example, $\lambda = i$ and $f_1(x) \neq 0$. Indeed, the left-hand side is real and the right-hand side is imaginary.

(iii) Indeed, for any $x \in V$, we have $f_1(ix) = \operatorname{Re} f(ix) = \operatorname{Re}(if(x)) = -f_2(x)$.

(iv) Linear functional $f_1 \colon V \to \mathbb{R}$ is dominated by p on V. Indeed, $|f_1(x)| = |\operatorname{Re} f(x)| \leq |f(x)| \leq p(x)$. By the real Hahn–Banach theorem, there exists $F_1 \colon X \to \mathbb{R}$, a linear functional on X over $\mathbb{R}$, such that $F_1 = f_1$ on V and $F_1 \leq p$ on X. As p is a seminorm (recall that a sublinear function which is additionally absolute homogeneous is a seminorm), it is $-F_1(x) = F_1(-x) \leq p(-x) = p(x)$, which shows, together with $F_1(x) \leq p(x)$, that $|F_1| \leq p$ on X.

(v) For an arbitrary $x \in X$, let $F(x) := F_1(x) - iF_1(ix)$. It is readily verified, directly from the definition, that F is a linear functional on X over $\mathbb{C}$. It is also an extension of f. Indeed, for $x \in V$, it is $F(x) = F_1(x) - iF_1(ix) = f_1(x) - if_1(ix) = f(x)$. It remains to verify that $|F|$ is dominated by p. Let $x \in X$ be arbitrary and fixed. There exists $t \in \mathbb{R}$ such that $|F(x)| = e^{it}F(x) = F(e^{it}x) = F_1(e^{it}x) - iF_1(ie^{it}x)$. The left-hand side is real and F_1 is real-valued so it must be $|F(x)| = F_1(e^{it}x) \leq p(e^{it}x) = |e^{it}|p(x) = p(x)$.

Thus we have proved the complex version of the Hahn–Banach theorem:

Corollary. *Let X be a complex vector space and $V \subset X$ be its subspace. Let $f \colon V \to \mathbb{C}$ be linear, $p \colon X \to \mathbb{R}$ be a seminorm, and $|f| \leq p$ on V. Then there exists a linear $F \colon X \to \mathbb{C}$ such that $F = f$ on V and $|F| \leq p$ on X.*

Problem 7.6 (Mazur's lemma). Let X be a real vector space and $M \subset X$ be an arbitrary set. We define the *convex hull* of M as

$$\begin{aligned}\operatorname{conv} M &:= \{x \in X,\ x \text{ is a finite convex combination of elements of } M\} \\ &= \left\{\begin{matrix} x \in X, \text{ there exists } m \in \mathbb{N}, \text{ positive numbers } \lambda_1, \dots, \lambda_m \text{ with} \\ \sum_{j=1}^m \lambda_j = 1, \text{ and vectors } x_1, \dots, x_m \in M \text{ such that } x = \sum_{j=1}^m \lambda_j x_j \end{matrix}\right\}.\end{aligned}$$

(i) Show that $M \subset \operatorname{conv} M$, that $\operatorname{conv} M$ is convex, and that $\overline{\operatorname{conv} M}$ is convex.

(ii) Use Theorem 7.2 to prove the following result:

Theorem (Mazur's lemma). *Let X be a real normed space and suppose that $\{x_j\}_{j=1}^\infty \subset X$ converges weakly to some $x \in X$. Then $x \in \overline{\operatorname{conv}\{x_j\}_{j=1}^\infty}$.*

(iii) Show that this statement is equivalently formulated as follows:

Theorem (Mazur's lemma). *Let X be a real normed space and suppose that $\{x_j\}_{j=1}^\infty \subset X$ converges weakly to some $x \in X$. Then there exists a sequence of finite convex combinations of $\{x_j\}_{j=1}^\infty$ which converges strongly to x. Precisely, there exists a sequence of integers $\{m_j\}_{j=1}^\infty$ and numbers $0 \leq \lambda_{ji} \leq 1$, $j = 1, 2, \dots$, $i = 1, 2, \dots, m_j$, with $\sum_{i=1}^{m_j} \lambda_{ji} = 1$ for every $j \in \mathbb{N}$, such that*

$$\sum_{i=1}^{m_j} \lambda_{ji} x_i \to x \qquad \text{strongly as } j \to \infty.$$

Solution.

(ii) $M \coloneqq \overline{\operatorname{conv}\{x_j\}_{j=1}^\infty}$ is a closed convex subset of a normed space, so by Theorem 7.2 it is weakly closed. Hence, for $\{x_j\}_{j=1}^\infty$, a weakly convergent sequence from M, its limit must be in M. ◻

Week 8.

PROPOSITION 8.1. *A closed subspace of a reflexive normed space is reflexive.*

Proof. Suppose that V is a reflexive normed space (and hence Banach) and $M \subset V$ is a closed subspace. We need to show that the canonical map $J\colon M \to M^{**}\colon x \mapsto Jx$ given by

$$(Jx)(\varphi) = \varphi(x) \qquad \text{for all } \varphi \in M^*$$

is surjective. Equivalently, for arbitrary given $f \in M^{**}$ we need to find $x \in M$ such that $Jx = f$, i.e.,

$$\varphi(x) = f(\varphi) \qquad \text{for all } \varphi \in M^*. \tag{8.1}$$

Fix $f \in M^{**}$. Define a linear function F on V^* by

$$F(\Phi) = f(\Phi_{|M}) \qquad \text{for } \Phi \in V^*.$$

We leave it as an exercise to verify that $F \in V^{**}$. As V is reflexive, there is $x \in V$ such that

$$\Phi(x) = F(\Phi) = f(\Phi_{|M}) \qquad \text{for all } \Phi \in V^*.$$

We claim that $x \in M$. For contradiction assume that $x \in V \setminus M$. As M is closed, by the Hahn–Banach theorem there is $\Psi \in V^*$ with $\Psi(x) = 1$ and $\Psi = 0$ on M. Hence

$$1 = \Psi(x) = F(\Psi) = f(\Psi_{|M}) = f(0) = 0,$$

which is the desired contradiction and thus $x \in M$. It remains to show that (8.1) is satisfied for given $f \in M^{**}$ and here constructed $x \in M$. Fix $\varphi \in M^*$. With its Hahn–Banach extension $\Phi \in V^*$, $\Phi_{|M} = \varphi$, we have

$$f(\varphi) = f(\Phi_{|M}) = F(\Phi) = \Phi(x) = \Phi_{|M}(x) = \varphi(x). \qquad ◻$$

PROBLEM 8.2 (weak convergence in $C(K)$). Let $f \in C([0,1])$ be piecewise affine such that

$$f_n(0) = f_n(\tfrac{1}{n+2}) = f_n(\tfrac{1}{n}) = f_n(1) = 0, \qquad f_n(\tfrac{1}{n+1}) = 1.$$

Show that f_n does not converge in norm and that f_n converges to zero pointwise everywhere in $[0,1]$.

QUIZ 8.3. Consider the sequence $\{g_n\}_{n=1}^\infty$ of functions $g_n\colon [0,1] \to \mathbb{R}$ given by

$$g_n(x) = \exp(nx).$$

For arbitrary $z \in [0,1]$, consider the Dirac functional $\delta_z\colon C([0,1]) \to \mathbb{R}$,

$$\delta_z(f) = f(z).$$

Which of the following possibilities are true?

(A) δ_z is a linear functional;

(B) δ_z is a bounded linear functional;
(C) number sequence $\{\delta_z(g_n)\}_{n=1}^{\infty}$ converges for $z = \frac{1}{2}$;
(D) the sequence $\{g_n\}_{n=1}^{\infty}$ converges weakly in $C([0, 1])$.

We can see that a weakly convergent sequence from $C([0, 1])$ is (i) pointwise convergent in $[0, 1]$ and (ii) bounded.[1] In fact, (i) and (ii) together are sufficient conditions for weak convergence in $C([0, 1])$:

PROPOSITION 8.4. *Let K be a compact metric space. Then $\{f_n\}_{n=1}^{\infty}$ converges weakly in $C(K)$ if and only if $\{\|f_n\|_\infty\}$ is bounded and $\{f_n\}$ converges pointwise everywhere in K.*

THEOREM 8.5 (F. Riesz representation theorem). *Let K be a compact metric space and suppose that F is a non-negative linear functional on $C(K)$, i.e., if $h \in C(K)$ and $h \geq 0$ on K then $F(h) \geq 0$. There exists a Radon measure μ on K such that*

$$F(h) = \int_K h \, \mathrm{d}\mu \qquad \text{for every } h \in C(K).$$

Such measure is unique on Borel subsets of K.

Proof of Proposition 8.4. Above we have seen the proof of "⇒", so suppose w.l.o.g. that $f_n \to 0$ everywhere in K and $\sup_{n,x} |f_n(x)| < \infty$. Fix $F \in \big(C(K)\big)^*$ arbitrary. We need to show that $F(f_n) \to 0$ as $n \to \infty$. By Theorem 8.5 there is a real-valued Radon measure μ on K with $F(f_n) = \int_K f_n \, \mathrm{d}\mu$. The integral goes to zero by the dominated convergence theorem (f_n is dominated by a constant function and $f_n \to 0$ pointwise). ∎

Proposition 8.4 shows that $\{f_n\}$ from Problem 8.2 converges weakly to zero, while it does not converge in norm.

PROBLEM 8.6. Consider Lagrange interpolation at arbitrary nodes $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$, i.e., for $f \in C([-1, 1])$, there is a unique $L_n f \in \mathcal{P}_n$ such that $(L_n f)(x_j) = f(x_j)$, $j = 0, \ldots, n$. Such $L_n f$ is given by

$$L_n f = \sum_{j=0}^{n} f(x_j) \ell_j, \qquad \ell_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}.$$

(i) Show that $\{L_n\}_{n=1}^{\infty}$ are linear bounded operators on $C([-1, 1])$ and that $\|L_n\| = \| \sum_{j=1}^{n} |\ell_j| \|_\infty$.

Function $\sum_j |\ell_j|$ is called the Lebesgue function (for nodes $x_0, \ldots, x_n$) and number $\|L_n\|$ is called the Lebesgue constant (for nodes $x_0, \ldots, x_n$). Whatever the choice of the nodes is, the Lebesgue constant $\|L_n\|$ grows at least logarithmically:

THEOREM (Faber, Fejér). *If $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$, then $\|L_n\| \geq \frac{1}{12} \log n$.*

(ii) Use the theorem and the uniform boundedness principle (Banach–Steinhaus theorem) to show that there exists $f \in C([0, 1])$ such that $\sup_n \|L_n f\| = \infty$.
(iii) Conclude that for such f, the sequence $L_n f$ does not converge in $\| \cdot \|_\infty$. In particular $L_n f$ does not converge to f.

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[1] PROPOSITION FROM THE CLASS (a consequence of the uniform boundedness principle). Weakly convergent sequences in a normed space are bounded.

Homework 8.7. Let U be a Banach space and let $T\colon U \to \ell_\infty$ be a linear operator defined on whole U, i.e., such that $Tx \in \ell_\infty$ for every $x \in U$. Consider its components $T_j\colon U \to \mathbb{R}$ given by $T_j(x) = (Tx)_j$ for $x \in U$ and $j \in \mathbb{N}$. Prove that T is bounded if and only if T_j, $j \in \mathbb{N}$, are all bounded. (Use the uniform boundedness principle to prove one of the implications.)

Solution. Suppose that T_j are all bounded and $Tx \in \ell_\infty$ for every $x \in U$. Thus for an arbitrary $x \in U$ we have

$$\infty > \|Tx\|_\infty = \sup_{j\in\mathbb{N}} |(Tx)_j| = \sup_{j\in\mathbb{N}} |T_j x|,$$

which shows that $\{T_j x\}_{j\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$.

As $\{T_j\}_{j\in\mathbb{N}}$ is a sequence of bounded linear operators from U to $\mathbb{R}$ and $\{T_j x\}_{j\in\mathbb{N}} \subset \mathbb{R}$ is a bounded sequence for every $x \in U$, the uniform boundedness principle yields that $\{T_j\}_{j\in\mathbb{N}}$ is a bounded sequence of operators. Hence

$$\infty > \sup_{j\in\mathbb{N}} \|T_j\| = \sup_{j\in\mathbb{N}} \sup_{\|x\|_U=1} |T_j x| = \sup_{\|x\|_U=1} \sup_{j\in\mathbb{N}} |T_j x| = \sup_{\|x\|_U=1} \|Tx\|_\infty = \|T\|.$$

Thus T is bounded. The proof of the opposite implication goes along the same lines but it does not require the uniform boundedness principle. ∎

Homework 8.8.

(i) Show that there exists a bounded linear functional F on ℓ_∞ such that $F(x) = \lim_{k\to\infty} x_k$ whenever x is a convergent sequence. (Use the Hahn–Banach extension theorem.)

(ii) Show that there exists a bounded linear functional $F \in L^\infty(\mathbb{R})^*$ such that $F(f) = \operatorname{ess\,lim}_{x\to 0} f(x)$ whenever the limit exists. (Use the Hahn–Banach extension theorem.)

(iii) Show that (ii) fails when $L^\infty(\mathbb{R})$ is replaced by $L^1(\mathbb{R})$. To do this, find a bounded sequence $\{f_n\}_{n=1}^\infty \subset L^1(\mathbb{R})$ with $F(f_n) \to \infty$ as $n \to \infty$.

Week 9.

Problem 9.1 (Everywhere-defined unbounded operator on a Banach space). Let X be an infinite dimensional vector space. We say that a set $M = \{v_i\}_{i\in I}$ is *linearly independent* if for every *finite* index set $J \subset I$, the equation $\sum_{j\in J} c_j v_j = 0$ implies that $c_j = 0$ for all $j \in J$. We say that a set $B \subset X$ is a *Hamel basis* of X if B is linearly independent and every element of X can be written as a *finite* linear combination of elements of B.

(i) Let a linearly independent sequence $\{b_i\}_{i=0}^\infty \subset X$ be given. Show, using Zorn's lemma, that there exists a Hamel basis B containing $\{b_i\}_{i=0}^\infty$ as its subset.

> "The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?" (Jerry Bona)

(ii) Let X be a Banach space. Recall that, by the Baire category theorem, $\{b_i\}_{i=0}^{\infty}$ alone cannot be a Hamel basis of X. In the other words, the dimension of X is uncountably infinite.

(iii) Now assume w.l.o.g. that $\|b_i\| = 1$ for $i = 1, 2, \dots$ and consider the function $F\colon B \to \mathbb{R}$ given as $F(b_i) = i$ for $i = 1, 2, \dots$ and $F(b) = 0$ for $b \in B \setminus \{b_i\}_{i=1}^{\infty}$. Show that F is uniquely extended to a *linear* functional $F\colon X \to \mathbb{R}$. Observe that F is unbounded.

Problem 9.2. Consider linear operators $T_n\colon \ell_2 \to \ell_2$ given by

$$T_n(x_1, x_2, \dots) = \big(x_1, \tfrac{x_2}{2}, \tfrac{x_3}{3}, \dots, \tfrac{x_n}{n}, 0, 0, \dots\big).$$

(i) For each $x \in \ell_2$, decide whether the pointwise limit of $\{T_n x\}_{n\in\mathbb{N}}$ exists, i.e., whether there is $Tx \in \ell_2$ such that

$$\|Tx - T_n x\| \to 0 \qquad \text{as } n \to \infty.$$

(ii) Use the uniform boundedness principle to argue that T defines a linear bounded operator.

(iii) Show that $\|T_n - T\| \to 0$ as $n \to \infty$ and conclude that T is compact.

(iv) Consider the sequence $\{e^n\}_{n\in\mathbb{N}} \subset \ell_2$, where $e^n := (0, \dots, 0, 1, 0, \dots)$ with 1 at n-th place. Show that $\{e^n\}$ does not converge in ℓ_2.

Proposition (Riesz representation theorem). *For every $F \in \ell_2^*$ there exists a unique $\{y_j\}_{j\in\mathbb{N}} \in \ell_2$ such that $F(x) = \sum_{j\in\mathbb{N}} y_j x_j$ for all $x = \{x_j\}_{j\in\mathbb{N}} \in \ell_2$.*

(v) For $F \in \ell^*$ consider the value $F(e^n)$. Use the proposition to show that $e_n \rightharpoonup 0$ weakly in ℓ_2.

(vi) Compute the norm limit of $\{Te^n\}_{n\in\mathbb{N}}$.

Problem 9.3. On the Banach space $\mathcal{C}([0,1])$ consider the following operators and decide whether they are compact linear operators.

(i) $Tf(x) = f(\cos^2(x))$,

(ii) $Tf(x) = \cos^2(f(x))$,

(iii) $Tf(x) = f(0)f'(x)$,

(iv) $Tf(x) = (x-1)xf(0) + \int_0^x f(s)\,\mathrm{d}s$,

(v) $Tf(x) = y(x)$, where y is the solution of the initial value problem $y' + y = f$ in $(0,1)$, $y(0) = 0$.

Solution.

(i) We will show that the closed unit ball $\bar{B}_1 \coloneqq \{f \in \mathcal{C}([0,1]), \|f\|_\infty \le 1\}$ is mapped by T onto a set containing $\bar{B}_1$. By the F. Riesz theorem (the closed unit ball in a Banach space is compact if and only if the space is finite-dimensional), $\bar{B}_1$ is not compact and thus T is not compact.
It remains to show that $T\bar{B}_1 \supset \bar{B}_1$. Let $g \in \bar{B}_1$ be arbitrary. We need to find $f \in \bar{B}_1$ such that $Tf = g$, i.e.,

$$f(\cos^2 x) = g(x) \qquad \text{for all } x \in [0,1]. \tag{9.1}$$

Obeserve that

$$f_1(y) = \begin{cases} g(1), & 0 \le y \le \cos^2 1, \\ g(\arccos\sqrt{y}), & \cos^2 1 \le y \le 1, \end{cases}$$

is one of the solutions of (9.1) and that $\|f_1\|_\infty \le \|g\|_\infty \le 1$ so that $f_1 \in \bar{B}_1$.

(ii, iii) The operators are clearly nonlinear.

(iv) The operator $f \mapsto (x-1)xf(0)$ is clearly linear, bounded, and of rank 1, hence compact.[2] So it remains to show that $f \mapsto \int_0^x f$ is compact. Let $\{f_n\}_{n=1}^\infty \subset \mathcal{C}([0,1])$ be a bounded sequence, i.e., $\|f_n\|_\infty \le M$. We need to show that $\{\int f_n\}_n$ is a compact sequence in $\mathcal{C}([0,1])$, i.e., that one can select a subsequence such that $\{\int f_{n_k}\}_k$ converges uniformly. By the Arzelà–Ascoli theorem (Theorem 3.2) this is the case if and only if $\int f_n$ is uniformly bounded and uniformly equicontinuous. Boundedness has been showed in Problem 4.4 (iv) and for the required continuity we can estimate

$$\left|\int_0^x f_n - \int_0^y f_n\right| \le \left|\int_x^y f_n\right| \le \|f_n\|_\infty |x-y| \le M|x-y|,$$

which shows that $\{\int f_n\}_n$ is a uniformly equicontinuous sequence.

(v) It has been showed in Problem 4.4 (v) that T is linear bounded and that there is an explicit formula for T,

$$(Tf)(x) = \int_0^x f(t)\exp(t-x)\,\mathrm{d}t.$$

Similarly as in (iv), take a bounded sequence $\{f_n\}_n \subset \mathcal{C}([0,1])$, $\|f_n\|_\infty \le 1$, and show that $\{Tf_n\}_n$ is uniformly equicontinuous and bounded. Then the Arzelà–Ascoli theorem (Theorem 3.2) yields compactness of T. ∎

Homework 9.4 (Baire property and meager sets in $L^p(\Omega)$). Let X be a topological space. Show that the following properties are equivalent.

(B1) Every countable union of closed sets with empty interior has empty interior.

(B2) Every countable intesection of dense open sets is dense.

We say that a set is a *nowhere dense subset of* X if its closure has empty interior. We say that a subset of X is a *meager subset of* X, *meager in* X, or of the *first category in* X if it is a countable union of nowhere dense subsets of X. A subset of X which is not meager in X is called a *nonmeager subset of* X, *nonmeager in* X, or of the *second category in* X. Then the Baire property (B1), (B2) is equivalently expressed as follows.

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[2] It is a common mistake to conclude that a linear (not necessarily bounded) operator with a finite rank is compact. As a counterexample consider the unbounded functional $F\colon X \to \mathbb{R}$ on an infinite-dimensional Banach space X constructed in Problem 9.1. Take arbitrary nonzero $y \in X$. Then $T\colon X \to X\colon x \mapsto F(x)y$ is defined on the whole X, has rank 1, and is certainly not compact (verify directly using the sequential definition of compactness).

(B3) Every meager subset of X has empty interior.
(B4) Every nonempty open subset of X is nonmeager in X.

(i) Show that the unit ball in $L^2((0,1))$, i.e., the set $\{f \in L^2((0,1)),\ \|f\|_2 < 1\}$, is a nowhere dense subset of $L^1((0,1))$.
(ii) Building on (i), decide whether $L^2((0,1))$ is a meager or nonmeager subset of $L^1((0,1))$.

Week 10.

HOMEWORK 10.1 (compactness of integral operator). Let $K\colon [a,b]\times[a,b] \to \mathbb{R}$ be a continuous function. Show that the integral operator $T\colon \mathcal{C}([a,b]) \to \mathcal{C}([a,b])$ given by

$$(Tf)(x) = \int_a^b K(x,y)\, f(y)\,\mathrm{d}y \tag{10.1}$$

is compact. Use the Arzelà–Ascoli theorem (Theorem 3.2).

For $f \in \mathcal{C}([-1,1])$ consider the following boundary value problem:

$$-u'' = f \quad \text{in } (-1,1), \qquad u(-1) = u(1) = 0.$$

Show that the solution to this problem is unique and that it is represented by the formula

$$u(x) = \int_{-1}^{x} \frac{(1+y)(1-x)}{2} f(y)\,\mathrm{d}y + \int_{x}^{1} \frac{(1-y)(1+x)}{2} f(y)\,\mathrm{d}y.$$

Show that the solution operator $f \mapsto u$ can be written in the form (10.1) with certain K and hence it is compact.

PROBLEM 10.2 (compact embedding of Hölder spaces). Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $0 < \alpha < \beta \le 1$ be given. Show that $C^{0,\beta}(\bar\Omega)$ is compactly embedded in $C^{0,\alpha}(\bar\Omega)$, i.e., show that the identity mapping from $C^{0,\beta}(\bar\Omega)$ to $C^{0,\alpha}(\bar\Omega)$ is compact. Use the Arzelà–Ascoli theorem (Theorem 3.2).

Consider the sequence $\{f_n\}$ from Problem 8.2 and set $u_n := \frac{1}{n^2} f_n$. Show that $\{u_n\}$ is bounded in $C^{0,1}([0,1])$, i.e., that for some M, $C > 0$, it is

$$\|u_n\|_\infty \le M \qquad \text{and} \qquad |u_n(x) - u_n(y)| \le C|x-y|,$$

and that $\{u_n\}$ does not converge in $C^{0,1}([0,1])$. Observe that $\{u_n\}$ converges in $C([0,1])$.

PROBLEM 10.3 (separability of ℓ_p and $L^p(\Omega)$).

(i) Show that every subset of a separable metric space is separable.
(ii) Show that ℓ_p is separable for every $1 \le p < \infty$ (cf. Homework 2.4 (i)) and that ℓ_∞ is not separable.
(iii) Let $\Omega \subset \mathbb{R}^d$ be open. Show that $L^p(\Omega)$ is separable for every $1 \le p < \infty$ and that, provided Ω is nonempty, $L^\infty(\Omega)$ is not separable.

PROBLEM 10.4 (dual of L^p). Let $\Omega \subset \mathbb{R}^d$ be open. Let $p \in (1,\infty)$ and $1/p + 1/p' = 1$. In the sequel we will use the notation $L^p := L^p(\Omega)$ and $(L^p)^* := (L^p(\Omega))^*$ for any $1 < p < \infty$. Consider the mapping $T\colon L^{p'} \to (L^p)^*$ given by

$$\langle Tu, f\rangle = \int_\Omega uf\,\mathrm{d}x, \qquad f \in L^p.$$

(i) Show that T is linear.

(ii) Show that T is isometry; precisely $\|Tu\|_{(L^p)^*} = \|u\|_{p'}$ for every $u \in L^{p'}$.

(iii) Show that $T(L^{p'})$, the range of T, is closed in $(L^p)^*$.

(iv) Show that $T(L^{p'})$ is dense in $(L^p)^*$. Use reflexivity of $L^{p'}$ and the following proposition.

LEMMA 10.5. *Let V be a normed space and $M \subset V$ be its subspace. Then $\overline{M} = V$ if and only if*

$$\{F \in V^*,\ F = 0 \text{ on } M\} = \{F \in V^*,\ F = 0 \text{ on } V\}.$$

(v) Conclude that, for $1 < p < \infty$, $(L^p)^*$ is isometrically isomorphic to $L^{p'}$ (through T).

Proof of Lemma 10.5. Suppose that M is dense. If $F = 0$ on M and $\{x_k\} \subset M$ is such that $x_k \to x \in \overline{M} = V$, then $0 = F(x_k) \to F(x)$ by virtue of continuity of F. As $x \in V$ was arbitrary, this shows that $F = 0$ on V.

For the opposite implication, suppose that $\overline{M}$ is a proper subspace of V. We use the following proposition from the class:

THEOREM (consequence of the Hahn–Banach theorem). *Let M be a closed proper subspace of a normed space V and let $x \in V \setminus M$ be given. Then there exists $F \in V^*$ such that $F = 0$ on M, $\|F\| = 1$, and $F(x) = \operatorname{dist}(x, M) > 0$.*

Thus, there exists a nonzero F that vanishes on $\overline{M}$, and, in particular, on M. ☐

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