FUNCTIONAL ANALYSIS FOR PHYSICISTS: EXERCISE PROBLEMS

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Week 1.

PROBLEM 1.1. Let $A \in \mathbb{R}^{n\times n}$ be given. The following assertions are equivalent:

(i) A is non-singular (the equation $Ax = b$ has one and only one solution for each $b \in \mathbb{R}^n$);
(ii) the mapping $x \mapsto Ax$ is injective (the equation $Ax = b$ has at most one solution for each $b \in \mathbb{R}^n$);
(iii) the mapping $x \mapsto Ax$ is surjective (the equation $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$).

In the following exercise we shall demonstrate that in the infinite-dimensional case (ii) and (iii) are not any more equivalent.

Consider mapping $T\colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ given by prescription

$$T\colon f(x) \mapsto f(x^2), \qquad x \in [0,1].$$

(i) Verify that this is a correct definition and that the mapping T is linear.
(ii) Show that $T - \mathrm{Id}$ is not injective.
(iii) Show that $T + \mathrm{Id}$ is injective.
(iv) Show that $T + \mathrm{Id}$ is not surjective.

Solution.

(ii) As T is linear, it is sufficient to show that there is a non-trivial solution of the homogeneous equation $(T - \mathrm{Id})f = 0$. This is indeed the case, as any constant function, e.g., $f \equiv 1$, is a solution.
(iii) Analogously, to show injectivity of $T + \mathrm{Id}$, we have to show that the only solution of the homogenous equation $(T + \mathrm{Id})f = 0$ is the zero function. Using the equation repeatedly, we obtain

$$f(x) = -f(x^2) = f(x^4) = -f(x^8) = f(x^{16}) = \cdots .$$

The first equality in particular implies that $f(0) = f(1) = 0$. By induction, for a fixed $a \in (0,1)$, we have, for any $n \in \mathbb{N}$, that

$$f(a) = (-1)^n f(a^{2^n}) \to 0 \qquad \text{as } n \to \infty,$$

with the limit due to continuity of f. This shows that $f(a) = 0$. As a was arbitrary from $(0,1)$, we conclude that $f \equiv 0$.
(iv) To show that $T + \mathrm{Id}$ is not surjective, we need to show that there exists $g \in \mathcal{C}([0,1])$ such that the equation $(T + \mathrm{Id})f = g$ does not have a solution $f \in \mathcal{C}([0,1])$. Assume there is a solution. We have

$$f(x^2) = g(x) - f(x), \qquad x \in [0,1],$$

which yields, with change of variable,

$$f(x) = g(x^{1/2}) - f(x^{1/2}), \qquad x \in [0,1],$$

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and, after recursive application of the equation,

$$f(x) = g(x^{1/2}) - g(x^{1/4}) + f(x^{1/4}), \qquad x \in [0,1],$$
$$\vdots$$
$$f(x) = \sum_{j=1}^{n} (-1)^{j-1} g(x^{2^{-j}}) + (-1)^n f(x^{2^{-n}}), \qquad x \in [0,1]. \tag{1.1}$$

Set $a \coloneqq 1/2$ and suppose that $g\colon [0,1] \to \mathbb{R}$ is a piecewise affine function interpolating the values

$$\begin{aligned} g(0) &\coloneqq 0, \\ g(a^{2^{-j}}) &\coloneqq \frac{(-1)^{j-1}}{j} \qquad \text{for } j \in \mathbb{N}, \\ g(1) &\coloneqq 0. \end{aligned}$$

It is left as a homework to show that $g \in \mathcal{C}([0,1])$. Substituting this choice of g into (1.1) yields, for $x \coloneqq a$,

$$f(a) = \sum_{j=1}^{n} \frac{1}{j} + (-1)^n f(a^{2^{-n}}).$$

The left-hand side is supposed to be a finite number by the required continuity of f, the first term on the right-hand side diverges as $n \to \infty$, and the last term goes to zero, which is the desired contradiction. □

Problem 1.2.

(i) For a $p \geq 1$ consider the set of sequences

$$\ell_p \coloneqq \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{R},\ \textstyle\sum_{k>0} |x_k|^p < \infty \right\}.$$

What is the relation between ℓ_p and ℓ_q given $1 \leq p < q < \infty$?

(ii) Let $\Omega \coloneqq (0,1)$. For a given $p \geq 1$ consider the set of p-integrable functions

$$L^p(\Omega) \coloneqq \left\{ u\colon \Omega \to \mathbb{R} \text{ measurable},\ \textstyle\int_\Omega |u|^p < \infty \right\}.$$

What is the relation between $L^p(\Omega)$ and $L^q(\Omega)$ given $1 \leq p < q < \infty$?

(iii) What is the relation between $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ given $1 \leq p < q < \infty$?

Solution.

(i) Let $\{y_k\}_{k=1}^{\infty}$ be arbitrary such that $\sum_k |y_k|^p = 1$. Then $|y_k| \leq 1$ for all $k \in \mathbb{N}$ and hence

$$\sum_{k\in\mathbb{N}} |y_k|^q \leq \sum_{k\in\mathbb{N}} |y_k|^p = 1. \tag{1.2}$$

Now for an arbitrary nonzero $x \in \ell_p$, set $y \coloneqq \frac{x}{(\sum |x_k|^p)^{1/p}}$, which satisfies $\sum_k |y_k|^p = 1$, and hence (1.2) can be used for this y. After little rearrangement one gets $(\sum_k |x_k|^q)^{1/q} \leq (\sum_k |x_k|^p)^{1/p}$, which proves the inclusion $\ell_p \subset \ell_q$.

(ii) Hölder's inequality, for $r \geq 1$,

$$\int_\Omega |fg| \leq \Big(\int_\Omega |f|^r\Big)^{1/r} \Big(\int_\Omega |g|^s\Big)^{1/s}, \qquad \frac{1}{r} + \frac{1}{s} = 1,$$

gives for $f := |u|^p$, $g := 1$, and $r := q/p$

$$\int_\Omega |u|^p \leq \Big(\int_\Omega |u|^q\Big)^{p/q} |\Omega|^{1-p/q}.$$

After rearrangement,

$$\Big(\int_\Omega |u|^p\Big)^{1/p} \leq |\Omega|^{1/p-1/q} \Big(\int_\Omega |u|^q\Big)^{1/q},$$

which shows that $L^q(\Omega) \subset L^p(\Omega)$ whenever $|\Omega| < \infty$.

(iii) For $\Omega = \mathbb{R}$ the above argument does not work and clearly there are functions from $L^p(\mathbb{R})$ which are not in $L^q(\mathbb{R})$ and vice versa. For $u(x) := \Xi_{(0,1)} x^{-1/p+\varepsilon}$, where Ξ_M denotes the characteristic function of set $M \subset \mathbb{R}$, it is $L^p(\mathbb{R}) \ni u \notin L^q(\mathbb{R})$ if $\varepsilon > 0$ is chosen sufficiently small. On the other hand, for $v(x) := \Xi_{(1,\infty)} x^{-1/q-\varepsilon}$ with $\varepsilon > 0$ sufficiently small, it is $L^p(\mathbb{R}) \not\ni v \in L^q(\mathbb{R})$. ◻

Week 2.

Problem 2.1. Decide which of the following are normed spaces. If so, determine whether they are Banach.

(i) $(\mathbb{R}^3, \|\cdot\|_{1/2})$ for

$$\|x\|_{1/2} = \Big(\sum_{j=1}^{3} |x_j|^{1/2}\Big)^2.$$

(ii) $(\mathbb{R}, \|\cdot\|_t)$ for

$$\|x\|_t = \begin{cases} 3x & \text{if } x \geq 0, \\ -x & \text{otherwise.} \end{cases}$$

(iii) The space of polynomials of degree at most 2 with

$$\|p\| := |p(1)| + |p'(1)| + \tfrac{1}{2}|p''(1)|$$

(iv) The space of all polynomials with the maximum norm $\|p\|_\infty = \max\limits_{x\in[0,1]} |p(x)|$.

Solution. (iv) The normed space $(\mathcal{P}, \|\cdot\|_\infty)$ of all polynomials on $[0,1]$ is not complete. The sequence of polynomials $\sum_{j=0}^{n} x^j/j!$, $n = 1, 2, \ldots$ converges uniformly in $[0,1]$, i.e., in the $\|\cdot\|_\infty$ norm, to $\exp(x) \notin \mathcal{P}$. ◻

Problem 2.2.

(i) Show that every subspace of a normed space is also a normed space (under the same norm).

(ii) Show that every closed subspace of a Banach space is also a Banach space (under the same norm).

Denote by ℓ_∞ the set of all bounded sequences of real or complex numbers, c the set of all convergent sequences of real or complex numbers, c_0 the set of all null (convergent to zero) sequences, and c_{00} the set of all eventually zero sequences (sequences with finitely many nonzero elements). Consider the supremum norm $\|x\|_\infty \coloneqq \sup_{k>0} |x_k|$ and show that

(iii) $(\ell_\infty, \|\cdot\|_\infty)$ is a Banach space,
(iv) c is a closed subspace of $(\ell_\infty, \|\cdot\|_\infty)$,
(v) c_0 is a closed subspace of $(c, \|\cdot\|_\infty)$, and
(vi) c_{00} is a subspace of $(c_0, \|\cdot\|_\infty)$ which is not closed.

Solution.

(iii) We leave the task to verify that $(\ell_\infty, \|\cdot\|_\infty)$ is a normed space for the reader and proceed with completeness. Suppose that $\{x^n\}_{n=1}^\infty \subset \ell_\infty$ is a Cauchy sequence, i.e., for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|x^m - x^n\|_\infty < \varepsilon$ for all $m, n > N$, or equivalently, using the definition of $\|\cdot\|_\infty$,

$$|x_k^n - x_k^m| < \varepsilon \qquad \text{for all } m, n > N \text{ and all } k \in \mathbb{N}. \tag{2.1}$$

In particular, for a fixed $k \in \mathbb{N}$ the number sequence $\{x_k^n\}_{n=1}^\infty \subset \mathbb{R}$ is Cauchy and hence convergent to $x_k \coloneqq \lim_{n\to\infty} x_k^n$. Taking the limit $m \to \infty$ in (2.1) yields that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_k^n - x_k| \le \varepsilon \qquad \text{for all } n > N \text{ and all } k \in \mathbb{N}, \tag{2.2}$$

which can be rewritten as $\|x^n - x\|_\infty \to 0$ as $n \to \infty$ where $x \coloneqq \{x_k\}_{k=1}^\infty$. Let us finish by verifying that $x \in \ell_\infty$. Indeed, fixing $\epsilon > 0$ arbitrarily, (2.2) implies that for some $N \in \mathbb{N}$

$$\big||x_k| - |x_k^{N+1}|\big| \le \varepsilon \qquad \text{for all } k \in \mathbb{N},$$

and in turn $|x_k| \le |x_k^{N+1}| + \varepsilon$ for all $k \in \mathbb{N}$. As $x^{N+1} \in \ell^\infty$ and ε is fixed, one immediatelly gets that $x \in \ell_\infty$.

(iv) Let us show the closedeness. Suppose that $\{x^n\}_{n=1}^\infty \subset c$ is a convergent sequence (in the $\|\cdot\|_\infty$ norm), i.e., $\|x^n - x\|_\infty \to 0$ as $n \to \infty$ and $x \in \ell_\infty$ due to its completeness. We shall show that $x \in c$. Let us fix $\varepsilon > 0$ to an arbitrary value. By the uniform convergence $x^n \to x$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_k^n - x_k| < \frac{\varepsilon}{3} \qquad \text{for all } n \ge N_\varepsilon \text{ and all } k \in \mathbb{N}.$$

The number sequence $\{x_k^{N_\varepsilon}\}_{k=1}^\infty$ is convergent by the hypothesis $x^{N_\varepsilon} \in c$, i.e., (for the above chosen $\varepsilon > 0$) there exists $K \in \mathbb{N}$ such that

$$|x_k^{N_\varepsilon} - x_\ell^{N_\varepsilon}| < \frac{\varepsilon}{3} \qquad \text{for all } k, \ell > K.$$

Altogether, for arbitrary $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$|x_k - x_\ell| \le |x_k - x_k^{N_\varepsilon}| + |x_k^{N_\varepsilon} - x_\ell^{N_\varepsilon}| + |x_\ell^{N_\varepsilon} - x_\ell| < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon$$

for all $k, \ell > K$. In the other words, the number sequence $\{x_k\}_{k=1}^\infty$ is Cauchy and hence $x \in c$.

(v) Let us show the closedeness. Suppose that $\{x^n\}_{n=1}^\infty \subset c_0$ is a convergent sequence (in the $\|\cdot\|_\infty$ norm), i.e., $\|x^n - x\|_\infty \to 0$ as $n \to \infty$ and $x \in c$ as $(c, \|\cdot\|_\infty)$ is a Banach space by virtue of the previous task (iv). We shall show that $x \in c_0$. Let us fix $\varepsilon > 0$ to an arbitrary value. By the uniform convergence $x^n \to x$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_k^n - x_k| < \frac{\varepsilon}{2} \qquad \text{for all } n \geq N_\varepsilon \text{ and all } k \in \mathbb{N}.$$

The number sequence $\{x_k^{N_\varepsilon}\}_{k=1}^\infty$ is null (convergent to zero) by the hypothesis $x^{N_\varepsilon} \in c_0$, i.e., (for the above chosen $\varepsilon > 0$) there exists $K \in \mathbb{N}$ such that

$$|x_k^{N_\varepsilon}| < \frac{\varepsilon}{2} \qquad \text{for all } k > K.$$

Altogether, for arbitrary $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that

$$|x_k| \leq |x_k - x_k^{N_\varepsilon}| + |x_k^{N_\varepsilon}| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

for all $k > K$. In the other words, the number sequence $\{x_k\}_{k=1}^\infty$ is null and hence $x \in c_0$.

(vi) The sequence $\{(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots)\}_{n=1}^\infty \subset c_{00}$ converges in the supremum norm to $(1, \frac{1}{2}, \frac{1}{3}, \dots) \in c_0$, which is not an element of c_{00}. Hence c_{00} is not closed in $(c_0, \|\cdot\|_\infty)$. ∎

Homework 1.

(i) Show that, for a fixed $p \in [1, \infty)$, c_{00} is dense in the Banach space $(\ell_p, \|\cdot\|_p)$, where

$$\|x\|_p = \Big(\sum_{j=1}^\infty |x_i|^p\Big)^{\frac{1}{p}}.$$

(ii) Show that the closure of c_{00} in the supremum norm $\|\cdot\|_\infty$ coincides with c_0.

Homework 2. We say a subset V of a metric space is (sequentially) *compact* if every sequence in V has a convergent subsequence with the limit in V.

Let X be a Banach space, a set $A \subset X$ be closed, and a set $B \subset X$ be compact. Show that the set $A + B \coloneqq \{x + y,\, x \in A,\, y \in B\}$ is closed in X.

Homework 3. Let

$$f_n(x) \coloneqq \begin{cases} \frac{1}{n} & \text{if } x \in (0, n), \\ 0 & \text{otherwise.} \end{cases}$$

For every $p \in [1, \infty]$, determine whether $\{f_n\}$ has a limit in $(L^p(\mathbb{R}), \|\cdot\|_p)$,

$$\|f\|_p = \Big(\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x\Big)^{\frac{1}{p}}, \qquad p \in [1, \infty),$$

$$\|f\|_\infty = \operatorname{ess\,sup}_{\mathbb{R}} |f(x)|.$$

Homework 4. Consider X, the set of continuous functions $u\colon [0, \infty) \to \mathbb{R}$ such that

$$\|u\|_e \coloneqq \sup_{x \in [0, \infty)} e^x |u(x)|$$

is finite. Show that $(X, \|\cdot\|_e)$ is a normed space and determine whether it is complete.

Week 3.

PROBLEM 3.1.

(i) Consider $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$, the vector space of continuous functions on $[0,1]$ equipped with the maximum norm $\|u\|_\infty := \max_{x\in[0,1]} |u(x)|$. Think through that this is a normed space. Show that it is complete.

(ii) Show that $(\mathcal{C}([0,1]), \|\cdot\|_1)$, $\|u\|_1 := \int_0^1 |u(x)| \, \mathrm{d}x$ is a normed space which is not complete. As a counterexample consider the sequence

$$f_n(x) := \begin{cases} 0, & x \le \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}(x - \frac{1}{2}) + \frac{1}{2}, & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n}, \\ 1, & \frac{1}{2} + \frac{1}{n} \le x. \end{cases}$$

(iii) ARZELÀ–ASCOLI THEOREM. *Let a sequence of continuous functions* $\{f_n\}_{n=1}^\infty \subset \mathcal{C}([0,1])$ *be given.*

If $\{f_n\}_{n=1}^\infty$ *is* uniformly bounded, *i.e., there exists* $M > 0$ *such that*

$$\|f_n\|_\infty \le M,$$

and uniformly equicontinuous, *i.e., for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $x, y \in [0,1]$ *with* $|x - y| < \delta$ *it holds*

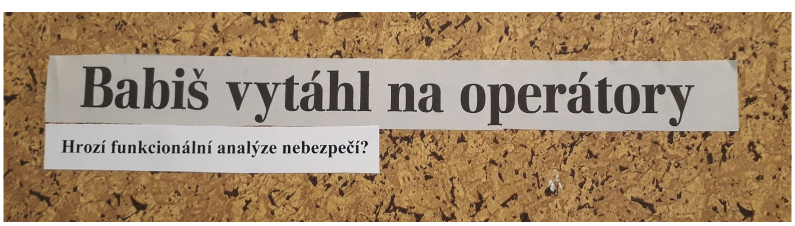
$$\sup_{n\in\mathbb{N}} |f_n(x) - f_n(y)| \le \varepsilon,$$

then there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ *that converges uniformly on* $[0,1]$.

The converse is true as well in the following sense: If every subsequence of $\{f_n\}_{n=1}^\infty$ *admits a uniformly convergent subsequence then* $\{f_n\}_{n=1}^\infty$ *is uniformly bounded and uniformly equicontinuous.*

Use the theorem to judge whether $\{f_n\}_{n=1}^\infty$ from (ii) is uniformly convergent.

Solution. (i) We leave this up to the reader. The $\varepsilon/3$ trick from Problem 2.2 (iv) can be used. □



PROBLEM 3.2.

(i) Let $A \in \mathbb{R}^{m\times n}$ be a given matrix. Consider the mapping $T_A \colon \mathbb{R}^m \to \mathbb{R}^n \colon x \mapsto Ax$. Verify that T_A is a linear bounded operator w.r.t. the Euclidean norm on $\mathbb{R}^m$ and $\mathbb{R}^n$. Does the operator norm $\|T_A\|$ coincide with some matrix norm of A? Is the norm attained for some $x \in \mathbb{R}^m$?

(ii) **(Diagonal operator on ℓ_p).** Let an arbitrary sequence $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{R}$ and $p \in [1,\infty]$ be given. Consider the operator $T : \ell_p \to \ell_p$ given by

$$T(x_1, x_2, x_3, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \ldots).$$

Equip ℓ_p with its usual norm $\|x\|_p \coloneqq \left(\sum_{k=1}^{\infty} |x|^p\right)^{1/p}$. Compute the norm of $T : (\ell_p, \|\cdot\|_p) \to (\ell_p, \|\cdot\|_p)$. When is the operator bounded?

(iii) For real functions on $[0,1]$, consider the differentiation mapping $f \mapsto f'$. This is clearly a linear operator. Consider the sequence $\{f_n\}_{n=1}^{\infty}$, $f_n(x) = \sin(nx)$. Compute $\|f_n\|_\infty$ and $\|f_n'\|_\infty$. Is the operator $(\mathcal{C}^1([0,1]), \|\cdot\|_\infty) \to (\mathcal{C}([0,1]), \|\cdot\|_\infty)\colon f \mapsto f'$ bounded?

(iv) **(Shift operator on L^p).** Let $a \in \mathbb{R}$ and $p \in [1,\infty]$ be given. Consider the mapping T_a given for a $f \in L^p(\mathbb{R})$ by prescription

$$(T_a f)(x) = f(x-a) \qquad \text{for a.e. } x \in \mathbb{R}.$$

Clearly T_a is a linear operator and $\|T_a f\|_p = \|f\|_p$. Hence, $T_a\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded with $\|T_a\| = 1$. Observe that T_a is a bijection.

(v) **(Shift operators on ℓ_p).** For any $1 \le p \le \infty$, define the *right shift* $S_R\colon \ell_p \to \ell_p$ and the *left shift* $S_L\colon \ell_p \to \ell_p$ by

$$S_R(x_1, x_2, x_3, \ldots) \coloneqq (0, x_1, x_2, \ldots),$$
$$S_L(x_1, x_2, x_3, \ldots) \coloneqq (x_2, x_3, x_4, \ldots).$$

Verify that these are bounded linear operators, compute their norms, and check whether they are injective or surjective.

(vi) **(Multiplication operator).** Let $\Omega \subset \mathbb{R}$ be open and let $g \in L^\infty(\Omega)$ be given. Consider the *multiplication operator*, which, for an $f \in L^p(\Omega)$, $1 \le p \le \infty$, is given by

$$(M_g f)(x) = f(x)\, g(x) \qquad \text{for a.e. } x \in \mathbb{R}.$$

Compute the norm of $M_g\colon L^p(\Omega) \to L^p(\Omega)$.

(vii) Consider the indefinite integral operator, for $f \in \mathcal{C}([a,b])$, $a < b$, given by

$$Tf(x) = \int_a^x f(s)\,\mathrm{d}s \qquad \text{for all } x \in [a,b].$$

Show that $T\colon (\mathcal{C}([a,b]), \|\cdot\|_\infty) \to (\mathcal{C}([a,b]), \|\cdot\|_\infty)$ is bounded and that $\|T\| = b-a$.

Do you know how can be the range of $T\colon L^1((a,b)) \to \mathcal{C}([a,b])$ described?

Week 4.

Problem 4.1. On the Banach space $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$ consider the following operators and decide whether they are linear and bounded:

(i) $Tf(x) = f(\cos^2(x))$,
(ii) $Tf(x) = \cos^2(f(x))$,
(iii) $Tf(x) = f(0)f'(x)$,
(iv) $Tf(x) = (x-1)xf(0) + \int_0^x f(s)\,\mathrm{d}s$,
(v) $Tf(x) = y(x)$, where y is the solution of the initial value problem $y' + y = f$ in $(0,1)$, $y(0) = 0$.

Solution.

(i) T is clearly linear and also bounded. Indeed, for arbitrary $x \in [0,1]$, it is

$$|f(\cos^2 x)| \le \max_{t\in[0,1]} |f(t)| = \|f\|_\infty.$$

Hence $\|Tf\|_\infty = \max_{x\in[0,1]} |f(\cos^2 x)| \le \|f\|_\infty$, which shows that $\|T\| \le 1$. Choosing $f \equiv 1$ shows that $\|T\| = 1$.

(ii) T is clearly non-linear.

(iii) T is clearly non-linear.

(iv) T is linear and, for arbitrary $x \in [0,1]$,

$$|Tf(x)| \le |f(0)|\,|x-1|\,|x| + \left|\int_0^x f(s)\,\mathrm{d}s\right| \le \tfrac14 |f(0)| + \int_0^1 |f(s)|\,\mathrm{d}s \\ \le \tfrac14 \|f\|_\infty + \|f\|_\infty.$$

Hence $\|T\| \le \frac54$ and T is bounded.

(v) For $f_1,\, f_2 \in \mathcal{C}([0,1])$, consider $y_1,\, y_2 \in \mathcal{C}([0,1])$ such that

$$\begin{aligned} y_1' + y_1 &= f_1 \qquad \text{in } (0,1), & y_1(0) &= 0,\\ y_2' + y_2 &= f_2 \qquad \text{in } (0,1), & y_2(0) &= 0. \end{aligned}$$

Due to the linearity of the equations, we have

$$(y_1+y_2)' + (y_1+y_2) = (f_1+f_2) \qquad \text{in } (0,1), \qquad (y_1+y_2)(0) = 0,$$

which shows that $T(f_1+f_2) = Tf_1 + Tf_2$. Proceeding similarly for homogeneity, we get that T is linear.

It is readily verified that T has the explicit representation

$$Tf(x) = \int_0^x \exp(t-x)\, f(t)\,\mathrm{d}t.$$

Hence, for any $x \in [0,1]$,

$$|Tf(x)| \le \int_0^x \exp(t-x)\,|f(t)|\,\mathrm{d}t \le \int_0^x |f(t)|\,\mathrm{d}t \le \|f\|_\infty.$$

Hence, T is bounded with $\|T\| \le 1$. ◻

Problem 4.2 (inequality used in [1, proof of Lemma 2.24]). Let $f\colon [0,\infty) \to \mathbb{R}$ be concave such that $f(0) \ge 0$. Show that then $f(a+b) \le f(a) + f(b)$ for all $a,\, b \ge 0$.

Solution. By hypotheses, we have, with $t \in [0,\infty)$ and $0 \le \lambda \le 1$, that

$$f(\lambda t) = f\big(\lambda t + (1-\lambda)0\big) \ge \lambda f(t) + (1-\lambda) f(0) \ge \lambda f(t).$$

Hence,

$$\begin{aligned} f(a) + f(b) &= f\Big(\frac{a}{a+b}(a+b)\Big) + f\Big(\frac{b}{a+b}(a+b)\Big)\\ &\ge \frac{a}{a+b} f(a+b) + \frac{b}{a+b} f(a+b) = f(a+b). \end{aligned}$$ ◻

Example 4.3 (examples of Fréchet spaces [1, Examples 2.25, 2.26]).

Week 5.

Example 5.1 (Schwartz space of rapidly decreasing functions [2]). The Schwartz space (the space of rapidly decreasing functions)

$$\mathcal{S}(\mathbb{R}^n) := \big\{ u \in C^\infty(\mathbb{R}^n),\ \|x^\beta \partial_\alpha u\|_\infty < \infty \text{ for all multiindices } \alpha,\, \beta \big\}$$

is a Fréchet space (without proof) when equipped with the sequence of seminorms $\{p_j\}_{j=0}^{\infty}$,

$$p_j(u) \coloneqq \sum_{|\alpha|,|\beta|\le j} \|x^\beta \partial_\alpha u\|_\infty,$$

or, for example, $\{q_j\}_{j=0}^{\infty}$,

$$q_j(u) \coloneqq \max_{|\alpha|\le j} \|(1+|x|^2)^j \partial_\alpha u\|_\infty.$$

These two generate the same topology. Significance of the space is that (i) Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is one-to-one, (ii) Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n)' \to \mathcal{S}(\mathbb{R}^n)'$ on tempered distributions $\mathcal{S}(\mathbb{R}^n)'$ is naturally defined (by moving $\mathcal{F}$ to test functions), and (iii) as $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, $\mathcal{F}$ can be extended to $\hat{\mathcal{F}}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which is unitary. For details see [2].

Problem 5.2 (Minkowski functional). Let X be a real normed space and $B \subset X$ be a non-empty convex open set containing the origin. Let the functional $p : X \to [0,\infty)$ be defined by

$$p(x) \coloneqq \inf\{\lambda > 0,\, x \in \lambda B\}, \qquad \text{for every } x \in X.$$

Show that

(i) there exists $M > 0$ such that $p(x) \le M\|x\|$ for all $x \in X$;
(ii) $B = \{x \in X,\, p(x) < 1\}$;
(iii) p is sublinear, i.e.,

$$\begin{aligned} p(\alpha x) &= \alpha p(x) && \text{for all } x \in X \text{ and } \alpha \ge 0 \text{ and} \\ p(x+y) &\le p(x) + p(y) && \text{for all } x,\, y \in X. \end{aligned}$$

Solution.

(i) By the hypothesis, there exists a ball $B_r \coloneqq \{x \in X,\, \|x\| < r\}$ with certain $r > 0$ such that $B_r \subset B$. Hence

$$p(x) = \inf\{\lambda > 0,\, \tfrac{x}{\lambda} \in B\} \le \inf\{\lambda > 0,\, \tfrac{x}{\lambda} \in B_r\} = \frac{\|x\|}{r}.$$

(ii) To show "$\subset$", suppose that $x \in B$. As B is open, $(1+\delta)x \in B$ for some $\delta > 0$ small enough. In the other words, $\frac{x}{\lambda} \in B$ for $\lambda = \frac{1}{1+\delta}$, and hence

$$p(x) = \inf\{\lambda > 0,\, \tfrac{x}{\lambda} \in B\} \le \inf\Big\{\frac{1}{1+\delta}\Big\} = \frac{1}{1+\delta} < 1.$$

For the opposite inclusion, suppose that $p(x) < 1$. By the definition of p, there exists $0 < \beta < 1$ such that $x/\beta \in B$. As B is convex and contains the origin, we have

$$x = \beta\,\frac{x}{\beta} + (1-\beta)\,0 \in B.$$

(iii) We leave the task to verify positive homogeneity, $p(\alpha x) = \alpha p(x)$, for all $x \in X$ and $\alpha \ge 0$, up to the reader, so it remains to prove the triangle inequality.

Suppose that $x, y \in X$ and fix $\varepsilon > 0$. Then for $\frac{x}{p(x)+\varepsilon}$, we have

$$p\left(\frac{x}{p(x)+\varepsilon}\right) = \frac{p(x)}{p(x)+\varepsilon} < 1,$$

where the equality follows from the positive homogeneity, and hence, by virtue of (ii), $\frac{x}{p(x)+\varepsilon} \in B$. Similarly, $\frac{y}{p(y)+\varepsilon} \in B$. By the convexity of B, it follows that, with arbitrary $0 < \mu < 1$,

$$\mu \frac{x}{p(x)+\varepsilon} + (1-\mu) \frac{y}{p(y)+\varepsilon} \in B.$$

Chossing $\mu := \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$ and using (ii) and the absolute homogeneity yields

$$1 > p\left(\frac{x+y}{p(x)+p(y)+2\varepsilon}\right) = \frac{p(x+y)}{p(x)+p(y)+2\varepsilon}.$$

As ε was arbitrary, it is $p(x+y) \le p(x) + p(y)$. □

Homework 5 (Hahn–Banach separation theorem, weak topology). For a function $f\colon X \to \mathbb{R}$, its *epigraph* is defined as

$$\operatorname{epi} f := \{(x, y) \in X \times \mathbb{R},\ y \ge f(x)\}.$$

Lemma 5.1. *Suppose that $f\colon X \to \mathbb{R}$ is convex. Then $\operatorname{epi} f$ is convex.*

If X is a normed space, the product $X \times \mathbb{R}$ is a normed space with, e.g., $\|(x, y)\|_{X\times\mathbb{R}} := \|x\|_X + |y|$. Recall we say that a function $f\colon X \to \mathbb{R}$ is (norm) lower semicontinuous if $x_n \to x$ (in norm) implies $\liminf_{n\to\infty} f(x_n) \ge f(x)$.

Lemma 5.2. *Suppose that X is a normed space and $f\colon X \to \mathbb{R}$ is (norm) lower semicontinuous. Then $\operatorname{epi} f$ is (norm) closed.*

We say that a subset $M \subset X$ of a normed space X is *(sequentially) weakly closed* if every weakly convergent sequence $\{x_n\}_{n\ge1} \subset M$ satisfies $x_n \rightharpoonup x \in M$. We can immediately see that a weakly closed set is closed. Indeed, suppose that $\{x_n\} \subset M$ conveges in norm to $x \in X$. Then $\{x_n\}$ converges weakly to the same x. As M is weakly closed, it is necessarily $x \in M$. The converse holds true for convex sets:

Lemma 5.3. *A subset of a real normed space that is closed and convex is weakly closed.*

We say that $f\colon X \to \mathbb{R}$ is *weakly lower semicontinuous* if the weak convergence $x_n \rightharpoonup x$ implies $\liminf_{n\to\infty} f(x_n) \ge f(x)$.

Theorem 5.4. *Let f be a functional on a real normed space which is lower semicontinuous and convex. Then f is weakly lower semicontinuous.*

Corollary 5.5. *Let V be a normed space (either real or complex). Then the norm $\|\cdot\|\colon V \to \mathbb{R}\colon x \mapsto \|x\|$ is weakly lower semicontinuous, i.e.,*

$$\liminf_{n\to\infty} \|x_n\| \ge \|x\| \qquad \textit{whenever } x_n \rightharpoonup x.$$

Prove the lemmas, the theorem, and the corollary. Lemma 5.3 can be proved by contradiction, invoking the Hahn–Banach (strict) separation theorem. (Recall that any singleton set is compact). The lemmas can all be proved independently. Argue carefully for proof of the corollary in the complex case.

Week 6.

PROBLEM 6.1 (complex Hahn–Banach theorem).

(i) Let V be a vector space over $\mathbb{C}$. Show that V is a vector space over $\mathbb{R}$.

(ii) Let $f\colon V \to \mathbb{C}$ be a linear functional on the complex vector space V. Define $f_1,\, f_2\colon V \to \mathbb{R}$ by

$$f_1(x) \coloneqq \operatorname{Re} f(x),$$
$$f_2(x) \coloneqq \operatorname{Im} f(x).$$

Show that f_1 and f_2 are linear functionals on V over $\mathbb{R}$, but they are not, in general, linear functionals on V over $\mathbb{C}$.

(iii) Show that $f_2(x) = -f_1(ix)$, and hence $f(x) = f_1(x) - if_1(ix)$.

(iv) Let X be a complex vector space, $p\colon X \to \mathbb{R}$ be a seminorm, and let $V \subset X$ be a subspace of X. Suppose that $f\colon V \to \mathbb{C}$ is linear such that $|f(x)| \le p(x)$ on V. Apply the real version of Hahn–Banach theorem to construct a linear $F_1\colon X \to \mathbb{R}$, an extension of $f_1\colon V \to \mathbb{R}$, such that $|F_1| \le p$ on X.

(v) From F_1 construct a linear $F\colon X \to \mathbb{C}$, an extension of $f\colon V \to \mathbb{C}$, and show that $|F| \le p$ on X.

Solution.

(ii) For arbitrary $x,\, y \in V$, we have $f_1(x+y) = \operatorname{Re} f(x+y) = \operatorname{Re} f(x) + \operatorname{Re} f(y) = f_1(x)+f_1(y)$. As of homogeneity, we have $f_1(\lambda x) = \operatorname{Re} f(\lambda x) = \operatorname{Re}(\lambda f(x))$ for any $\lambda \in \mathbb{C}$. If λ is real, then the last expression equals $\lambda f_1(x)$, which shows that f_1 is linear on V over $\mathbb{R}$. On the other hand, homogeneity $f_1(\lambda x) = \lambda f_1(x)$ is clearly violated if, for example, $\lambda = i$ and $f_1(x) \neq 0$. Indeed, the left-hand side is real and the right-hand side is imaginary.

(iii) Indeed, for any $x \in V$, we have $f_1(ix) = \operatorname{Re} f(ix) = \operatorname{Re}(if(x)) = -f_2(x)$.

(iv) Linear functional $f_1\colon V \to \mathbb{R}$ is dominated by p on V. Indeed, $|f_1(x)| = |\operatorname{Re} f(x)| \le |f(x)| \le p(x)$. By the real Hahn–Banach theorem, there exists $F_1\colon X \to \mathbb{R}$, a linear functional on X over $\mathbb{R}$, such that $F_1 = f_1$ on V and $F_1 \le p$ on X. As p is a seminorm (recall that a sublinear function which is additionally absolute homogeneous is a seminorm), it is $-F_1(x) = F_1(-x) \le p(-x) = p(x)$, which shows, together with $F_1(x) \le p(x)$, that $|F_1| \le p$ on X.

(v) For an arbitrary $x \in X$, let $F(x) \coloneqq F_1(x) - iF_1(ix)$. It is readily verified, directly from the definition, that F is a linear functional on X over $\mathbb{C}$. It is also an extension of f. Indeed, for $x \in V$, it is $F(x) = F_1(x) - iF_1(ix) = f_1(x) - if_1(ix) = f(x)$. It remains to verify that $|F|$ is dominated by p. Let $x \in X$ be arbitrary and fixed. There exists $t \in \mathbb{R}$ such that $|F(x)| = e^{it}F(x) = F(e^{it}x) = F_1(e^{it}x) - iF_1(ie^{it}x)$. The left-hand side is real and F_1 is real-valued so it must be $|F(x)| = F_1(e^{it}x) \le p(e^{it}x) = |e^{it}|p(x) = p(x)$.

Thus we have proved the complex version of the Hahn–Banach theorem:

COROLLARY. *Let X be a complex vector space and $V \subset X$ be its subspace. Let $f\colon V \to \mathbb{C}$ be linear, $p\colon X \to \mathbb{R}$ be a seminorm, and $|f| \le p$ on V. Then there exists a linear $F\colon X \to \mathbb{C}$ such that $F = f$ on V and $|F| \le p$ on X.*

PROBLEM 6.2 (Mazur's lemma). Let X be a real vector space and $M \subset X$ be an arbitrary set. We define the *convex hull* of M as

$$\begin{aligned}\operatorname{conv} M &\coloneqq \{x \in X,\ x \text{ is a finite convex combination of elements of } M\}\\ &= \left\{\begin{matrix} x \in X, \text{ there exists } m \in \mathbb{N}, \text{ positive numbers } \lambda_1, \dots, \lambda_m \text{ with} \\ \sum_{j=1}^m \lambda_j = 1, \text{ and vectors } x_1, \dots, x_m \in M \text{ such that } x = \sum_{j=1}^m \lambda_j x_j \end{matrix}\right\}.\end{aligned}$$

(i) Show that $M \subset \operatorname{conv} M$ and that $\operatorname{conv} M$ is convex.
(ii) Use Lemma 5.3 to prove the following result:

THEOREM (Mazur's lemma). *Let X be a real normed space and suppose that $\{x_j\}_{j=1}^\infty \subset X$ converges weakly to some $x \in X$. Then $x \in \overline{\operatorname{conv}\{x_j\}_{j=1}^\infty}$.*

(iii) Show that this statement is equivalently formulated as follows:

THEOREM (Mazur's lemma). *Let X be a real normed space and suppose that $\{x_j\}_{j=1}^\infty \subset X$ converges weakly to some $x \in X$. Then there exists a sequence of finite convex combinations of $\{x_j\}_{j=1}^\infty$ which converges strongly to x. Precisely, there exists a sequence of integers $\{m_j\}_{j=1}^\infty$ and numbers $0 \le \lambda_{ji} \le 1$, $j = 1, 2, \ldots$, $i = 1, 2, \ldots, m_j$, with $\sum_{i=1}^{m_j} \lambda_{ji} = 1$ for every $j \in \mathbb{N}$, such that*

$$\sum_{i=1}^{m_j} \lambda_{ji} x_i \to x \qquad \text{strongly as } j \to \infty.$$

Week 7.

PROBLEM 7.1 (on separability).
(i) Show that every subset of a separable metric space is separable.
(ii) Show that ℓ_p is separable for every $1 \le p < \infty$ and that ℓ_∞ is not separable.
(iii) Let $\Omega \subset \mathbb{R}^d$ be open. Show that $L^p(\Omega)$ is separable for every $1 \le p < \infty$ and that, provided Ω is not empty, $L^\infty(\Omega)$ is not separable.

PROBLEM 7.2 (Baire property). Let X be a topological space. Show that the following properties are equivalent.
(i) Every countable union of closed sets with empty interior has empty interior.
(ii) Every countable intesection of dense open sets is dense.
We say that a set is a *nowhere dense subset of* X if its closure has empty interior. We say that a subset of X is a *meager subset of* X, *meager in* X, or of the *first category in* X if it is a countable union of nowhere dense subsets of X. A subset of X which is not meager in X is called a *nonmeager subset of* X, *nonmeager in* X, or of the *second category in* X. Then the Baire property (i), (ii) is equivalently expressed as follows.
(iii) Every meager subset of X has empty interior.
(iv) Every nonempty open subset of X is nonmeager in X.

HOMEWORK 6.
(i) Show that the unit ball in $L^2((0,1))$, i.e., the set $\{f \in L^2((0,1)), \|f\|_2 < 1\}$, is a nowhere dense subset of $L^1((0,1))$.
(ii) Building on (i), decide whether $L^2((0,1))$ is a meager or nonmeager subset of $L^1((0,1))$.

PROBLEM 7.3 (Everywhere-defined unbounded operator on a Banach space). Let X be an infinite dimensional vector space. We say that a set $M = \{v_i\}_{i\in I}$ is *linearly independent* if for every *finite* index set $J \subset I$, the equation $\sum_{j\in J} c_j v_j = 0$ implies that $c_j = 0$ for all $j \in J$. We say that a set $B \subset X$ is a *Hamel basis* of X if B is linearly independent and every element of X can be written as a *finite* linear combination of elements of B.
(i) Let a linearly independent sequence $\{b_i\}_{i=0}^\infty \subset X$ be given. Show, using Zorn's lemma, that there exists a Hamel basis B containing $\{b_i\}_{i=0}^\infty$ as its subset.

(ii) Let X be a Banach space. Recall that, by the Baire category theorem, $\{b_i\}_{i=0}^{\infty}$ alone cannot be a Hamel basis of X. In the other words, the dimension of X is uncountably infinite.

(iii) Now assume w.l.o.g. that $\|b_i\| = 1$ for $i = 1, 2, \dots$ and consider the function $F\colon B \to \mathbb{R}$ given as $F(b_i) = i$ for $i = 1, 2, \dots$ and $F(b) = 0$ for $b \in B \setminus \{b_i\}_{i=1}^{\infty}$. Show that F is uniquely extended to a *linear* functional $F\colon X \to \mathbb{R}$. Show that F is unbounded.

HOMEWORK 7.

(i) Recall that for every vector space a Hamel basis exists by Zorn's lemma; cf. Problem 7.3 (i). Use the Baire category theorem to show that for every infinite-dimensional Banach space its Hamel basis is uncountable.

(ii) Consider $\mathcal{P}$, the space of polynomials in one variable of arbitrary degree with real coefficients. Show the following statement: There does not exist a functional $\|\cdot\|\colon \mathcal{P} \to \mathbb{R}$ such that $(\mathcal{P}, \|\cdot\|)$ is a Banach space.

HOMEWORK 8. Let U be a Banach space and let $T\colon U \to \ell_\infty$ be a linear operator defined on whole U, i.e., such that $Tx \in \ell_\infty$ for every $x \in U$. Consider its components $T_j\colon U \to \mathbb{R}$ given by $T_j(x) = (Tx)_j$ for $x \in U$ and $j \in \mathbb{N}$. Prove that T is bounded if and only if T_j, $j \in \mathbb{N}$, are all bounded.

HOMEWORK 9.

(i) Show that there exists a bounded linear functional F on ℓ_∞ such that $F(x) = \lim_{k\to\infty} x_k$ whenever x is a convergent sequence.

(ii) Show that there exists a bounded linear functional $F \in L^\infty(\mathbb{R})^*$ such that $F(f) = \operatorname{ess\,lim}_{x\to 0} f(x)$ whenever the limit exists.

(iii) Show that (ii) fails when $L^\infty(\mathbb{R})$ is replaced by $L^1(\mathbb{R})$. To do this, find a bounded sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R})$ with $F(f_n) \to \infty$ as $n \to \infty$.

Week 8.

PROBLEM 8.1 (dual of L^p). Let $\Omega \subset \mathbb{R}^d$ be open. Let $p \in (1, \infty)$ and $1/p + 1/p' = 1$. In the sequel we will use the notation $L^p := L^p(\Omega)$ and $(L^p)^* := (L^p(\Omega))^*$ for any $1 < p < \infty$. Consider the mapping $T\colon L^{p'} \to (L^p)^*$ given by

$$\langle Tu, f\rangle = \int_\Omega uf \,\mathrm{d}x, \qquad f \in L^p.$$

(i) Show that T is linear.

(ii) Show that T is isometry; precisely $\|Tu\|_{(L^p)^*} = \|u\|_{p'}$ for every $u \in L^{p'}$.

(iii) Show that $T(L^{p'})$, the range of T, is closed in $(L^p)^*$.

(iv) Show that $T(L^{p'})$ is dense in $(L^p)^*$. Use reflexivity of $L^{p'}$ and the following proposition.

LEMMA 8.1. *Let V be a normed space and $M \subset V$ be its subspace. Then $\overline{M} = V$ if and only if*

$$\{F \in V^*,\, F = 0 \text{ on } M\} = \{F \in V^*,\, F = 0 \text{ on } V\}.$$

(v) Conclude that, for $1 < p < \infty$, $(L^p)^*$ is isometrically isomorphic to $L^{p'}$ (through T).

Proof of Lemma 8.1. Suppose that M is dense. If $F = 0$ on M and $\{x_k\} \subset M$ is such that $x_k \to x \in \overline{M} = V$, then $0 = F(x_k) \to F(x)$ by virtue of continuity of F. As $x \in V$ was arbitrary, this shows that $F = 0$ on V.

For the opposite implication, suppose that $\overline{M}$ is a proper subspace of V. We use the following proposition from the class:

THEOREM 8.2 (a consequence of the Hahn–Banach theorem). *Let M be a closed proper subspace of a normed space V and let $x \in V \setminus M$ be given. Then there exists $F \in V^*$ such that $F = 0$ on M, $\|F\| = 1$, and $F(x) = \operatorname{dist}(x, M) > 0$.*

Thus, there exists a non-zero F that vanishes on $\overline{M}$, and, in particular, on M. ◻

Solution.

(ii) Given an arbitrary $u \in L^{p'}$, we obtain, using the definition of T and the Hölder inequality,

$$\|Tu\|_{(L^p)^*} = \sup_{f \in L^p} \frac{\langle Tu, f\rangle}{\|f\|_p} = \sup_{f \in L^p} \frac{\int u\, f}{\|f\|_p} \le \|u\|_{p'}.$$

On the other hand, the function $f_u := |u|^{p'-2}u$ belongs to L^p, which is easily verfied by checking that $(p'-1)p = p'$, and it is $\|f_u\|_p = \|u\|_{p'}^{p'-1}$. Hence

$$\|Tu\|_{(L^p)^*} = \sup_{f \in L^p} \frac{\int u\, f}{\|f\|_p} \ge \frac{\int u\, f_u}{\|f_u\|_p} = \frac{\|u\|_{p'}^{p'}}{\|u\|_{p'}^{p'-1}} = \|u\|_{p'}.$$

Altogether we have that $\|u\|_{p'} \le \|Tu\|_{(L^p)^*} \le \|u\|_{p'}$, which shows that the inequality is actually an equality.

(iv) Denote $E := T(L^{p'})$. To show that $\overline{E} = (L^p)^*$, it is sufficient (and necessary) by Lemma 8.1 to show that: if an arbitrary $h \in (L^p)^{**}$ vanishes on E then $h = 0$. Suppose that $h \in (L^p)^{**}$ vanishes on E, i.e., $\langle h, Tu\rangle_{(L^p)^{**},(L^p)^*} = 0$ for every $u \in L^{p'}$. As L^p is reflexive for $1 < p < \infty$, there exists $h \in L^p$ (denoted the same as $h \in (L^p)^{**}$) such that $\langle h, F\rangle_{(L^p)^{**},(L^p)^*} = \langle F, h\rangle_{(L^p)^*,L^p}$ for every $F \in (L^p)^*$. For $F := Tu$, we have $0 = \langle h, Tu\rangle_{(L^p)^{**},(L^p)^*} = \langle Tu, h\rangle_{(L^p)^*,L^p} = \int_\Omega u\, h$ for every $u \in L^{p'}$. The choice $u := |h|^{p-2}h$ is an admissible test function from $L^{p'}$, which is easily verified by checking that $(p-1)p' = p$. Thus $0 = \int_\Omega |h|^p = \|h\|_p^p$, hence $h \in L^p$ is the zero function, and, by the isometry of the canonical embedding, $h \in (L^p)^{**}$ is the zero functional.

(v) Isometry of T immediatelly implies that T is injective. As $E := T(L^{p'})$, the range of T, is closed and dense, it is $E = \overline{E} = (L^p)^*$. Hence T is surjective. ◻

PROBLEM 8.2 (compactness of integral operator). Let $K\colon [a,b]\times[a,b]\to\mathbb{R}$ be a continuous function. Show that the integral operator $T\colon \mathcal{C}([a,b])\to\mathcal{C}([a,b])$ given by

$$(Tf)(x) = \int_a^b K(x,y)\, f(y)\,\mathrm{d}y \tag{8.1}$$

is compact. Use the Arzelà–Ascoli theorem.

For $f\in\mathcal{C}([-1,1])$ consider the following boundary value problem:

$$-u'' = f \quad \text{in } (-1,1), \qquad u(-1)=u(1)=0.$$

Show that the solution to this problem is unique and that it is represented by the formula

$$u(x) = \int_{-1}^{x} \frac{(1+y)(1-x)}{2} f(y)\,\mathrm{d}y + \int_x^1 \frac{(1-y)(1+x)}{2} f(y)\,\mathrm{d}y.$$

Show that the solution operator $f\mapsto u$ can be written in the form (8.1) with certain K and hence it is compact.

Week 9.

Solution of Homework 8. The right implication is the easy one. We will prove the left one. Suppose that T_j are all bounded and $Tx\in\ell_\infty$ for all $x\in U$. Let $x\in U$ be arbitrary and fixed. Then

$$\infty > \|Tx\|_\infty = \sup_{j\in\mathbb{N}} |(Tx)_j| = \sup_{j\in\mathbb{N}} |T_j x|.$$

As $\{T_j\}_{j\in\mathbb{N}}$ are all bounded operators from U to $\mathbb{R}$ and $\{T_j x\}_{j\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$ for every $x\in U$, the uniform boundedness principle yields that $\{T_j\}_{j\in\mathbb{N}}$ is a bounded sequence of operators. Hence

$$\infty > \sup_{j\in\mathbb{N}} \|T_j\| = \sup_{j\in\mathbb{N}} \sup_{\|x\|_U=1} |T_j x| = \sup_{\|x\|_U=1} \sup_{j\in\mathbb{N}} |T_j x| = \sup_{\|x\|_U=1} \|Tx\|_\infty = \|T\|. \qquad \square$$

PROBLEM 9.1. Let $f\in\mathcal{C}([0,1])$, $q>1$, and $n\in\mathbb{N}$ be given. We consider the approximation problem of finding $p_*\in\mathcal{P}_n$, the space of polynomials of degree at most n, that would be the closest to f in the $L^q(0,1)$ norm. Denote

$$M \coloneqq \inf_{p\in\mathcal{P}_n} \|f-p\|_q.$$

(i) Let $q=2$. Use the projection theorem in Hilbert spaces to show that $M = \|f-p_*\|_2$ for some $p_*\in\mathcal{P}_n$, that p_* is uniquely given, and that the mapping $f\mapsto p_*$ is linear.

(ii) Let $q>1$ be arbitrary. Show that there exists a unique $p_*\in\mathcal{P}_n$ such that $M=\|f-p_*\|_q$ and that the mapping $f\mapsto p_*$ is nonlinear unless $q=2$.

Solution. (ii) We will present *the direct method in the calculus of variations.*

Step 1 (show that $M>-\infty$). Clearly it is $M\ge 0$. One also gets that $M<\infty$ as for the zero polynomial one gets $M\le\|f\|_q<\infty$.

Step 2 (take a minimizing sequence). This is trivial, from the definition of infimum: $M=\lim_{k\to\infty}\|f-p_k\|_q$ for some sequence $\{p_k\}_{k=1}^\infty\subset\mathcal{P}_n$.

Step 3 (establish a limit). If we show that $\{p_k\}_{k=1}^{\infty}$ is bounded in $L^q(0,1)$, the Banach–Alaoglu theorem and reflexivity of L^q assure that there is a weakly convergent subsequence, i.e., $p_{k_j} \rightharpoonup p_*$ weakly in L^q. So it remains to show the boundedness: $\|p_k\|_q \le \|p_k - f\|_q + \|f\|_q \to M + \|f\|_q < \infty$, i.e., the sequence $\|p_k\|_q$ is dominated by a convergent sequence.

Step 4 (show inclusion of the limit in the trial space). $\mathcal{P}_n$ is a finite-dimensional subspace of $L^q(0,1)$, hence closed and in turn, by Lemma 5.3, weakly closed. So the weak convergence $p_{k_j} \rightharpoonup p_*$ implies $p_* \in \mathcal{P}_n$.

Step 5 (pass to the limit in the functional). Our functional is $p \mapsto F(p) := \|f - p\|_q$. In this step one wants to show that $F(p_*) = M$. As $F(p_{k_j}) \to M$ by construction, this step amounts to showing that $F(p_{k_j}) \to F(p_*)$. So far we have established that p_{k_j} is weakly convergent. It is left as an exercise to show that $F\colon L^q(0,1) \to \mathbb{R}$ given above is continuous and convex. We can use Theorem 5.4 to deduce that F is weakly lower semicontinuous. Hence the weak convergence implies that $\liminf_{j\to\infty} F(p_{k_j}) \ge F(p_*)$. In the other words,

$$M = \lim_{k\to\infty} \|f - p_k\|_q = \liminf_{j\to\infty} \|f - p_{k_j}\|_q \ge \|f - p_*\|_q \ge M.$$

Both the left-hand side and the right-hand side are M, so we conclude that $M = \|f - p_*\|_q$.

Step 6 (uniqueness). Suppose that $\|f - p_1\|_q = \|f - p_2\|_q = M$ for distinct p_1, $p_2 \in \mathcal{P}_n$. Then, for arbitrary fixed $\lambda \in (0,1)$,

$$\begin{aligned} \|f - \lambda p_1 - (1-\lambda)p_2\|_q &= \|\lambda(f - p_1) + (1-\lambda)(f - p_2)\|_q \\ &< \lambda\|f - p_1\|_q + (1-\lambda)\|f - p_2\|_q = M, \end{aligned}$$

where the inequality follows from the strict convexity of $g \mapsto \|g\|_q$ (recall that $q > 1$), and this is a contradiction: $\|f - p_\lambda\| < \inf_{p\in\mathcal{P}_n} \|f - p\|$ for $p_\lambda := \lambda p_1 + (1-\lambda)p_2 \in \mathcal{P}_n$.

It remains to show nonlinearity of the projection $f \mapsto p_*$ for $q \ne 2$. ☐

Week 10.

Problem 10.1. On the Banach space $\mathcal{C}([0,1])$ consider the following operators and decide whether they are compact linear operators.

(i) $Tf(x) = f(\cos^2(x))$,
(ii) $Tf(x) = \cos^2(f(x))$,
(iii) $Tf(x) = f(0)f'(x)$,
(iv) $Tf(x) = (x-1)xf(0) + \int_0^x f(s)\,\mathrm{d}s$,
(v) $Tf(x) = y(x)$, where y is the solution of the initial value problem $y' + y = f$ in $(0,1)$, $y(0) = 0$.

Homework 10. Let $q > 1$, $f \in L^q(0,1)$, and $n \in \mathbb{N}$ be given. Consider functional $F : \mathcal{P}_n \to [0,\infty)$ given by

$$F(p) := \tfrac{1}{q}\|f - p\|_q^q.$$

(i) Compute the Gateaux derivative of F.
(ii) Formulate the necessary condition for $p_* = \operatorname{argmin}_{p\in\mathcal{P}_n} F(p)$.
(iii) Show that the necessary condition is sufficient.

(iv) Show that the mapping $P\colon L^q(0,1) \to \mathcal{P}_n\colon f \mapsto p_*$ is a projection onto $\mathcal{P}_n$, i.e., it is an idempotent mapping ($P^2 = P$) and the range of P is $\mathcal{P}_n$.

(v) Show that P is nonlinear unless $q = 2$.

Homework 11. Solve Problem 10.1 (i), (v).

Homework 12. Let $H := L^2(0,\pi)$ and $f_n(x) := \sin^2 nx$.

(i) Given $0 \le a \le b \le \pi$, compute $\int_0^\pi f_n(x)\Xi_{(a,b)}(x)\,\mathrm{d}x = \int_a^b f_n$, where $\Xi_{(a,b)}$ is the characteristic function of interval (a,b). Verify that the integral tends to $\frac{b-a}{2}$ as $n \to \infty$.

(ii) Show that $\int_0^\pi f_n(x)\varphi(x)\,\mathrm{d}x \to \int_0^\pi \frac12\varphi(x)\,\mathrm{d}x$ for every step function $\varphi\colon [0,\pi] \to \mathbb{R}$.

(iii) Recall that for an arbitrary $\varphi \in L^2(0,\pi)$ and $\varepsilon > 0$, the exists a step function φ_ε such that $\|\varphi - \varphi_\varepsilon\|_2 < \varepsilon$. Consider the identity

$$\begin{aligned}\int_0^\pi &\big(f_n(x) - \tfrac12\big)\varphi(x)\,\mathrm{d}x \\ &= \int_0^\pi \big(f_n(x) - \tfrac12\big)\big(\varphi(x) - \varphi_\varepsilon(x)\big)\,\mathrm{d}x - \int_0^\pi \big(f_n(x) - \tfrac12\big)\varphi_\varepsilon(x)\,\mathrm{d}x\end{aligned}$$

and the facts shown above to prove that $f_n \rightharpoonup \frac12$ weakly in $L^2(0,\pi)$.

(iv) Show that f_n does not have a strong limit (in $L^2(0,\pi)$).

(v) Consider linear operator $T\colon L^2(0,\pi) \to L^2(0,\pi)$ given by $Tf(x) = \int_0^x f(y)\,\mathrm{d}y$. This is a bounded linear operator. Indeed,

$$\begin{aligned}\|Tf\|_2^2 &= \int_0^\pi \Big|\int_0^x f(y)\,\mathrm{d}y\Big|^2\,\mathrm{d}x \\ &\le \int_0^\pi \Big(\int_0^x |f(y)|\,\mathrm{d}y\Big)^2\,\mathrm{d}x \le \int_0^\pi \Big(\int_0^\pi |f(y)|\,\mathrm{d}y\Big)^2\,\mathrm{d}x \le \pi^2\|f\|_2^2,\end{aligned}$$

where the last inequality follows from the Hölder inequality. Show that T is compact using Kolmogorov's criterion.

(vi) Compute Tf_n and $T\frac12$, show that $\|Tf_n - T\frac12\|_\infty \to 0$, and conclude that $Tf_n \to T\frac12$ strongly in $L^2(0,\pi)$.

Homework 13. Let Y be the subset of ℓ_2 given by

$$Y := \big\{x = \{x_i\}_{i=1}^\infty \in \ell_2,\ x_{2k-1} = x_{2k} \text{ for all integers } k \ge 1\big\}.$$

(i) Show that Y is a closed subspace of ℓ_2.

(ii) Identify the orthogonal complement of Y in ℓ_2.

(iii) Identify the ℓ_2-orthogonal projection onto Y.

Homework 14.

Theorem 10.1 (Hahn–Banach theorem in Hilbert spaces). *Let H be a Hilbert space, $Y \subset H$ be a subspace, and $f \in Y^*$ be an arbitrary linear bounded functional on Y. Then there exists a bounded linear functional $F \in H^*$ such that $F = f$ on Y and $\|F\| = \|f\|$. Besides, such F is unique.*

For the proof of this theorem, carry out the following steps. Notice that Zorn's lemma (axiom of choice) has not been invoked, in contrast to the Hahn–Banach theorem in nonseparable Banach spaces.

(i) Show that there exists a unique continuous extension of f from Y to $\overline{Y}$. If $y \in \overline{Y}$, then there exists $\{y_n\} \subset Y$ such that $y_n \to y$ in norm. Define $\hat{f}(y) \coloneqq \lim_{n\to\infty} f(y_n)$. Show that this is a correct definition (i.e, $y \mapsto \hat{f}(y)$ is a function), that $\hat{f}$ is linear, and that $\hat{f}$ is bounded. Show that such $\hat{f}$ is uniquely given.

(ii) Now for an arbitrary $x \in H$, consider its decomposition $x = Px + (I - P)x$, where P is the orthogonal projection onto $\overline{Y}$, cf. the direct sum theorem. Set $F(x) \coloneqq \hat{f}(Px)$ for all $x \in H$, verify that F is a linear extension of f, and compute the norm of F.

(iii) It remains to show uniqueness of F with such properties. This is postponed to the exercise session or is considered a bonus task.

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