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## FUNCTIONAL ANALYSIS FOR PHYSICISTS: EXERCISE PROBLEMS

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#### Week 1. 4

| 5        | PROBLEM 1.1. Let $A \in \mathbb{R}^{n \times n}$ be given. The following assertions are equivalent:  |
|----------|--|
| 6<br>7   | (i) A is non-singular (the equation $Ax = b$ has one and only one solution for each $b \in \mathbb{R}^n$ ):  |
| 8        | (ii) the mapping $x \mapsto Ax$ is injective (the equation $Ax = b$ has at most one  |
| 9        | solution for each $b \in \mathbb{R}^n$ );  |
| 10       | (iii) the mapping $x \mapsto Ax$ is surjective (the equation $Ax = b$ has at least one   |
| 11       | solution for each $b \in \mathbb{R}^n$ ).  |
| 12       | In the following exercise we shall demonstrate that in the infinite-dimensional case   |
| 13       | (ii) and (iii) are not any more equivalent. $T = \mathcal{L}([0, 1]) = \mathcal{L}([0, 1])$  |
| 14       | Consider mapping $T: \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ given by prescription  |
| 15       | $T \colon f(x) \mapsto f(x^2), \qquad x \in [0, 1].$   |
| 16       | (i) Verify that this is a correct definition and that the mapping $T$ is linear.   |
| 17       | (ii) Show that $T - \text{Id}$ is not injective.   |
| 18       | (iii) Show that $T + \text{Id}$ is injective.  |
| 19       | (iv) Show that $T + Id$ is not surjective.   |
| 20       | Solution.  |
| 21       | (ii) As $T$ is linear, it is sufficient to show that there is a non-trivial solution of  |
| 22       | the homogeneous equation $(T - \mathrm{Id})f = 0$ . This is indeed the case, as any  |
| 23       | constant function, e.g., $f \equiv 1$ , is a solution.   |
| 24<br>25 | (iii) Analogously, to snow injectivity of $I + Id$ , we have to snow that the only solution of the homogeneous equation $(T + Id)f = 0$ is the zero function |
| 20<br>26 | Using the equation repeatedly, we obtain   |
| 20       | come the equation repeatedly, we obtain  |
| 27       | $f(x) = -f(x^2) = f(x^4) = -f(x^8) = f(x^{16}) = \cdots$   |
| 28       | The first equality in particular implies that $f(0) = f(1) = 0$ . By induction,  |
| 29       | for a fixed $a \in (0, 1)$ , we have, for any $n \in \mathbb{N}$ , that  |
| 30       | $f(a) = (-1)^n f(a^{2^n}) \to 0$ as $n \to \infty$ ,   |
| 31       | with the limit due to continuity of f. This shows that $f(a) = 0$ . As a was   |
| 32       | arbitrary from $(0,1)$ , we conclude that $f \equiv 0$ .   |
| 33       | (iv) To show that $T + \text{Id}$ is not surjective, we need to show that there exists   |
| 34       | $g \in \mathcal{C}([0,1])$ such that the equation $(T + \mathrm{Id})f = g$ does not have a solution  |
| 35       | $f \in \mathcal{C}([0,1])$ . Assume there is a solution. We have   |
| 36       | $f(x^2) = g(x) - f(x), \qquad x \in [0, 1],$   |
| 37       | which yields, with change of variable,   |
| 38       | $f(x) = q(x^{1/2}) - f(x^{1/2}), \qquad x \in [0, 1],$   |
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$$f(x) = g(x^{1/2}) - g(x^{1/4}) + f(x^{1/4}), \qquad x \in [0, 1],$$

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$$\vdots$$
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and, after recursive application of the equation,

42 (1.1) 
$$f(x) = \sum_{j=1}^{n} (-1)^{j-1} g(x^{2^{-j}}) + (-1)^n f(x^{2^{-n}}), \quad x \in [0,1].$$

43 Set  $a \coloneqq 1/2$  and suppose that  $g: [0,1] \to \mathbb{R}$  is a piecewise affine function 44 interpolating the values

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$$g(0) \coloneqq 0,$$

46 
$$g(a^{2^{-j}}) \coloneqq \frac{(-1)^{j-1}}{j} \quad \text{for } j \in \mathbb{N},$$

$$47 g(1) \coloneqq 0.$$

It is left as a homework to show that 
$$g \in \mathcal{C}([0,1])$$
. Substituting this choice  
of  $g$  into  $(1.1)$  yields, for  $x \coloneqq a$ ,

50 
$$f(a) = \sum_{j=1}^{n} \frac{1}{j} + (-1)^n f(a^{2^{-n}}).$$

51 The left-hand side is supposed to be a finite number by the required continuity 52 of f, the first term on the right-hand side diverges as  $n \to \infty$ , and the last 53 term goes to zero, which is the desired contradiction.

54 **PROBLEM 1.2.** 

(i) For a  $p \ge 1$  consider the set of sequences

56 
$$\ell_p \coloneqq \{\{x_k\}_{k=1}^\infty \subset \mathbb{R}, \sum_{k>0} |x_k|^p < \infty\}.$$

57 What is the relation between  $\ell_p$  and  $\ell_q$  given  $1 \le p < q < \infty$ ?

58 (ii) Let  $\Omega \coloneqq (0,1)$ . For a given  $p \ge 1$  consider the set of *p*-integrable functions

$$L^p(\Omega) \coloneqq \left\{ u \colon \Omega \to \mathbb{R} \text{ measurable}, \int_{\Omega} |u|^p < \infty \right\}.$$

60 What is the relation between 
$$L^p(\Omega)$$
 and  $L^q(\Omega)$  given  $1 \le p < q < \infty$ ?

(iii) What is the relation between  $L^p(\mathbb{R})$  and  $L^q(\mathbb{R})$  given  $1 \le p < q < \infty$ ?

62 Solution.

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63 (i) Let  $\{y_k\}_{k=1}^{\infty}$  be arbitrary such that  $\sum_k |y_k|^p = 1$ . Then  $|y_k| \le 1$  for all  $k \in \mathbb{N}$ 64 and hence

65 (1.2) 
$$\sum_{k \in \mathbb{N}} |y_k|^q \le \sum_{k \in \mathbb{N}} |y_k|^p = 1.$$

66 Now for an arbitrary nonzero  $x \in \ell_p$ , set  $y \coloneqq \frac{x}{(\sum |x_k|^p)^{1/p}}$ , which satisfies 67  $\sum_k |y_k|^p = 1$ , and hence (1.2) can be used for this y. After little rearrange-68 ment one gets  $(\sum_k |x_k|^q)^{1/q} \leq (\sum_k |x_k|^p)^{1/p}$ , which proves the inclusion 69  $\ell_p \subset \ell_q$ .

71 
$$\int_{\Omega} |fg| \le \left( \int_{\Omega} |f|^r \right)^{1/r} \left( \int_{\Omega} |g|^s \right)^{1/s}, \qquad \frac{1}{r} + \frac{1}{s} = 1,$$

gives for  $f \coloneqq |u|^p$ ,  $g \coloneqq 1$ , and  $r \coloneqq q/p$ 

(ii) Hölder's inequality, for  $r \ge 1$ ,

$$\int_{\Omega} |u|^p \leq \left(\int_{\Omega} |u|^q\right)^{p/q} |\Omega|^{1-p/q}$$

74 After rearrangement,

(5) 
$$\left(\int_{\Omega} |u|^p\right)^{1/p} \le |\Omega|^{1/p-1/q} \left(\int_{\Omega} |u|^q\right)^{1/q},$$

76 which shows that  $L^q(\Omega) \subset L^p(\Omega)$  whenever  $|\Omega| < \infty$ .

(iii) For  $\Omega = \mathbb{R}$  the above argument does not work and clearly there are functions from  $L^p(\mathbb{R})$  which are not in  $L^q(\mathbb{R})$  and vice versa. For  $u(x) \coloneqq \Xi_{(0,1)} x^{-1/p+\varepsilon}$ , where  $\Xi_M$  denotes the characteristic function of set  $M \subset \mathbb{R}$ , it is  $L^p(\mathbb{R}) \ni$  $u \notin L^q(\mathbb{R})$  if  $\varepsilon > 0$  is chosen sufficiently small. On the other hand, for  $v(x) \coloneqq \Xi_{(1,\infty)} x^{-1/q-\varepsilon}$  with  $\varepsilon > 0$  sufficiently small, it is  $L^p(\mathbb{R}) \not\ni v \in L^q(\mathbb{R}). \square$ 

### 82 Week 2.

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PROBLEM 2.1. Decide which of the following are normed spaces. If so, determine
 whether they are Banach.

85 (i)  $(\mathbb{R}^3, \|\cdot\|_{1/2})$  for

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$$||x||_{1/2} = \left(\sum_{j=1}^{3} |x_j|^{1/2}\right)^2.$$

87 (ii)  $(\mathbb{R}, \|\cdot\|_t)$  for

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$$\|x\|_t = \begin{cases} 3x & \text{if } x \ge 0, \\ -x & \text{otherwise.} \end{cases}$$

89 (iii) The space of polynomials of degree at most 2 with

$$\|p\| := |p(1)| + |p'(1)| + \frac{1}{2}|p''(1)|$$

- 91 (iv) The space of all polynomials with the maximum norm  $||p||_{\infty} = \max_{x \in [0,1]} |p(x)|$ .
- 92 Solution. (iv) The normed space  $(\mathcal{P}, \|\cdot\|_{\infty})$  of all polynomials on [0, 1] is not 93 complete. The sequence of polynomials  $\sum_{j=0}^{n} x^j/j!$ ,  $n = 1, 2, \ldots$  converges 94 uniformly in [0, 1], i.e., in the  $\|\cdot\|_{\infty}$  norm, to  $\exp(x) \notin \mathcal{P}$ .
- 95 PROBLEM 2.2.
- (i) Show that every subspace of a normed space is also a normed space (under
   the same norm).
- (ii) Show that every closed subspace of a Banach space is also a Banach space
   (under the same norm).

100 Denote by  $\ell_{\infty}$  the set of all bounded sequences of real or complex numbers, c the set of 101 all convergent sequences of real or complex numbers,  $c_0$  the set of all null (convergent 102 to zero) sequences, and  $c_{00}$  the set of all eventually zero sequences (sequences with 103 finitely many nonzero elements). Consider the supremum norm  $||x||_{\infty} \coloneqq \sup_{k>0} |x_k|$ 

- 105 (iii)  $(\ell_{\infty}, \|\cdot\|_{\infty})$  is a Banach space,
- 106 (iv) c is a closed subspace of  $(\ell_{\infty}, \|\cdot\|_{\infty})$ ,
- 107 (v)  $c_0$  is a closed subspace of  $(c, \|\cdot\|_{\infty})$ , and
- 108 (vi)  $c_{00}$  is a subspace of  $(c_0, \|\cdot\|_{\infty})$  which is not closed.
- 109 Solution.
- (iii) We leave the task to verify that  $(\ell_{\infty}, \|\cdot\|_{\infty})$  is a normed space for the reader and proceed with completeness. Suppose that  $\{x^n\}_{n=1}^{\infty} \subset \ell_{\infty}$  is a Cauchy sequence, i.e., for every  $\varepsilon > 0$  there is  $N \in \mathbb{N}$  such that  $\|x^m - x^n\|_{\infty} < \varepsilon$  for all m, n > N, or equivalently, using the definition of  $\|\cdot\|_{\infty}$ ,

114 (2.1) 
$$|x_k^n - x_k^m| < \varepsilon$$
 for all  $m, n > N$  and all  $k \in \mathbb{N}$ .

115 In particular, for a fixed  $k \in \mathbb{N}$  the number sequence  $\{x_k^n\}_{n=1}^{\infty} \subset \mathbb{R}$  is Cauchy 116 and hence convergent to  $x_k \coloneqq \lim_{n \to \infty} x_k^n$ . Taking the limit  $m \to \infty$  in (2.1) 117 yields that for every  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that

118 (2.2) 
$$|x_k^n - x_k| \le \varepsilon$$
 for all  $n > N$  and all  $k \in \mathbb{N}$ ,

119 which can be rewritten as  $||x^n - x||_{\infty} \to 0$  as  $n \to \infty$  where  $x \coloneqq \{x_k\}_{k=1}^{\infty}$ . 120 Let us finish by verifying that  $x \in \ell_{\infty}$ . Indeed, fixing  $\epsilon > 0$  arbitrarily, (2.2) 121 implies that for some  $N \in \mathbb{N}$ 

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$$||x_k| - |x_k^{N+1}|| \le \varepsilon$$
 for all  $k \in \mathbb{N}$ ,

123 and in turn  $|x_k| \le |x_k^{N+1}| + \varepsilon$  for all  $k \in \mathbb{N}$ . As  $x^{N+1} \in \ell^{\infty}$  and  $\varepsilon$  is fixed, one 124 immediately gets that  $x \in \ell_{\infty}$ .

(iv) Let us show the closedeness. Suppose that  $\{x^n\}_{n=1}^{\infty} \subset c$  is a convergent sequence (in the  $\|\cdot\|_{\infty}$  norm), i.e.,  $\|x^n - x\|_{\infty} \to 0$  as  $n \to \infty$  and  $x \in \ell_{\infty}$ due to its completeness. We shall show that  $x \in c$ . Let us fix  $\varepsilon > 0$  to an arbitrary value. By the uniform convergence  $x^n \to x$ , there exists  $N_{\varepsilon} \in \mathbb{N}$ such that

130 
$$|x_k^n - x_k| < \frac{\varepsilon}{3}$$
 for all  $n \ge N_{\varepsilon}$  and all  $k \in \mathbb{N}$ .

131 The number sequence  $\{x_k^{N_{\varepsilon}}\}_{k=1}^{\infty}$  is convergent by the hypothesis  $x^{N_{\varepsilon}} \in c$ , i.e., 132 (for the above chosen  $\varepsilon > 0$ ) there exists  $K \in \mathbb{N}$  such that

133 
$$|x_k^{N_{\varepsilon}} - x_{\ell}^{N_{\varepsilon}}| < \frac{\varepsilon}{3} \quad \text{for all } k, \, \ell > K.$$

134 Altogether, for arbitrary  $\varepsilon > 0$  there exists  $K \in \mathbb{N}$  such that

135 
$$|x_k - x_\ell| \le |x_k - x_k^{N_\varepsilon}| + |x_k^{N_\varepsilon} - x_\ell^{N_\varepsilon}| + |x_\ell^{N_\varepsilon} - x_\ell| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all  $k, \ell > K$ . In the other words, the number sequence  $\{x_k\}_{k=1}^{\infty}$  is Cauchy and hence  $x \in c$ . 138 (v) Let us show the closedeness. Suppose that  $\{x^n\}_{n=1}^{\infty} \subset c_0$  is a convergent 139 sequence (in the  $\|\cdot\|_{\infty}$  norm), i.e.,  $\|x^n - x\|_{\infty} \to 0$  as  $n \to \infty$  and  $x \in c$ 140 as  $(c, \|\cdot\|_{\infty})$  is a Banach space by virtue of the previous task (iv). We shall 141 show that  $x \in c_0$ . Let us fix  $\varepsilon > 0$  to an arbitrary value. By the uniform 142 convergence  $x^n \to x$ , there exists  $N_{\varepsilon} \in \mathbb{N}$  such that

143 
$$|x_k^n - x_k| < \frac{\varepsilon}{2}$$
 for all  $n \ge N_{\varepsilon}$  and all  $k \in \mathbb{N}$ .

144 The number sequence  $\{x_k^{N_{\varepsilon}}\}_{k=1}^{\infty}$  is null (convergent to zero) by the hypothesis 145  $x^{N_{\varepsilon}} \in c_0$ , i.e., (for the above chosen  $\varepsilon > 0$ ) there exists  $K \in \mathbb{N}$  such that

146 
$$|x_k^{N_{\varepsilon}}| < \frac{\varepsilon}{2}$$
 for all  $k > K$ .

147 Altogether, for arbitrary  $\varepsilon > 0$  there exists  $K \in \mathbb{N}$  such that

148 
$$|x_k| \le |x_k - x_k^{N_{\varepsilon}}| + |x_k^{N_{\varepsilon}}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

- for all k > K. In the other words, the number sequence  $\{x_k\}_{k=1}^{\infty}$  is null and hence  $x \in c_0$ .
- 151 (vi) The sequence  $\{(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots)\}_{n=1}^{\infty} \subset c_{00}$  converges in the supre-152 mum norm to  $(1, \frac{1}{2}, \frac{1}{3}, \dots) \in c_0$ , which is not an element of  $c_{00}$ . Hence  $c_{00}$  is 153 not closed in  $(c_0, \|\cdot\|_{\infty})$ .

154 HOMEWORK 1.

(i) Show that, for a fixed  $p \in [1, \infty)$ ,  $c_{00}$  is dense in the Banach space  $(\ell_p, \|\cdot\|_p)$ , where

157 
$$||x||_p = \left(\sum_{j=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}.$$

(ii) Show that the closure of 
$$c_{00}$$
 in the supremum norm  $\|\cdot\|_{\infty}$  coincides with  $c_0$ .  
HOMEWORK 2. We say a subset V of a metric space is (sequentially) *compact* if  
every sequence in V has a convergent subsequence with the limit in V.

161 Let X be a Banach space, a set  $A \subset X$  be closed, and a set  $B \subset X$  be compact. 162 Show that the set  $A + B \coloneqq \{x + y, x \in A, y \in B\}$  is closed in X.

163 HOMEWORK 3. Let

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$$f_n(x) \coloneqq \begin{cases} \frac{1}{n} & \text{if } x \in (0, n), \\ 0 & \text{otherwise.} \end{cases}$$

165 For every  $p \in [1, \infty]$ , determine whether  $\{f_n\}$  has a limit in  $(L^p(\mathbb{R}), \|\cdot\|_p)$ ,

166 
$$||f||_p = \left(\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}, \qquad p \in [1,\infty),$$

167 
$$||f||_{\infty} = \operatorname{ess\,sup}_{\mathbb{R}} |f(x)|.$$

168 HOMEWORK 4. For each  $n \in \mathbb{N}$ , let the sequence  $\{x_k^n\}_{k=1}^{\infty} \subset \mathbb{R}$  be given by

169 
$$x_k^n = \frac{k+1}{k^2+2} + \frac{n+1}{n^2k}, \qquad k \in \mathbb{N}.$$

- (i) Determine whether  $x^n, n \in \mathbb{N}$ , belong to  $c_0, \ell_1, \ell_2, \ell_3$ , and  $\ell_{\infty}$ .
- (ii) Determine whether the sequence  $\{x^n\}_{n=1}^{\infty}$  converges in Banach spaces  $(c_0, \|\cdot\|_{172}$  $\|_{\infty})$  and  $(\ell_{\infty}, \|\cdot\|_{\infty})$ . If yes, establish the limit.

### 173 Week 3.

- 174 PROBLEM 3.1.
- (i) Consider  $(\mathcal{C}([0,1]), \|\cdot\|_{\infty})$ , the vector space of continuous functions on [0,1]equipped with the maximum norm  $\|u\|_{\infty} \coloneqq \max_{x \in [0,1]} |u(x)|$ . Think through that this is a normed space. Show that it is complete.
- (ii) Show that  $(\mathcal{C}([0,1]), \|\cdot\|_1), \|u\|_1 \coloneqq \int_0^1 |u(x)| \, dx$  is a normed space which is not complete. As a counterexample consider the sequence

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$$f_n(x) \coloneqq \begin{cases} 0, & x \le \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}(x - \frac{1}{2}) + \frac{1}{2}, & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n}, \\ 1, & \frac{1}{2} + \frac{1}{n} \le x. \end{cases}$$

- 181 (iii) ARZELÀ-ASCOLI THEOREM. Let a sequence of continuous functions  $\{f_n\}_{n=1}^{\infty}$ 182  $\subset C([0,1])$  be given.
- 183 If  $\{f_n\}_{n=1}^{\infty}$  is uniformly bounded, i.e., there exists M > 0 such that

$$\|f_n\|_{\infty} \le M$$

185 and uniformly equicontinuous, *i.e.*, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such 186 that for all  $x, y \in [0, 1]$  with  $|x - y| < \delta$  it holds

187 
$$\sup_{n \in \mathbb{N}} |f_n(x) - f_n(y)| \le \varepsilon,$$

188 then there exists a subsequence  $\{f_{n_k}\}_{k=1}^{\infty}$  that converges uniformly on [0, 1].

189 The converse is true as well in the following sense: If every subsequence 190 of  $\{f_n\}_{n=1}^{\infty}$  admits a uniformly convergent subsequence then  $\{f_n\}_{n=1}^{\infty}$  is uni-191 formly bounded and uniformly equicontinuous.

192 Use the theorem to judge whether  $\{f_n\}_{n=1}^{\infty}$  from (ii) is uniformly convergent. 193 Solution.

- (i) This is the *uniform limit theorem*. Its proof uses the  $\varepsilon/3$  strategy as in Problem 2.2 (iv).
  - (ii) The pointwise limit

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$$f(x) \coloneqq \begin{cases} 0, & x < \frac{1}{2}, \\ 1, & x > \frac{1}{2}, \end{cases}$$

198 does not belong to C([0, 1]), but a straightforward computation shows that 199  $||f_n - f||_1 \to 0 \text{ as } n \to \infty.$ 

(iii) Clearly it is  $||f_n|| \le 1$  for all  $n \in \mathbb{N}$ , so the sequence is uniformly bounded. On the other hand, the modulus of continuity blows up with  $n \to \infty$ : For arbitrary  $\varepsilon > 0$ , it is

$$|f_n(x) - f_n(y)| \le \varepsilon$$
 if  $|x - y| < \frac{2\varepsilon}{n}$ .



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Problem 3.2.

- (i) Let  $A \in \mathbb{R}^{n \times m}$  be a given matrix. Consider the mapping  $T_A : \mathbb{R}^m \to \mathbb{R}^n : x \mapsto Ax$ . Verify that  $T_A$  is a linear bounded operator w.r.t. the Euclidean norm on  $\mathbb{R}^m$  and  $\mathbb{R}^n$ . Does the operator norm  $||T_A||$  coincide with some matrix norm of A? Is the norm attained for some  $x \in \mathbb{R}^m$ ?
- (ii) (Diagonal operator on  $\ell_p$ ). Let an arbitrary sequence  $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$  and  $p \in [1, \infty]$  be given. Consider the operator  $T : \ell_p \to \ell_p$  given by

$$T(x_1, x_2, x_3, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \ldots)$$

.).

Equip  $\ell_p$  with its usual norm  $||x||_p \coloneqq \left(\sum_{k=1}^{\infty} |x|^p\right)^{1/p}$ . Compute the norm of  $T: (\ell_p, ||\cdot||_p) \to (\ell_p, ||\cdot||_p)$ . When is the operator bounded?

- (iii) For real functions on [0, 1], consider the differentiation mapping  $f \mapsto f'$ . This is clearly a linear operator. Consider the sequence  $\{f_n\}_{n=1}^{\infty}, f_n(x) = \sin(nx)$ . Compute  $||f_n||_{\infty}$  and  $||f'_n||_{\infty}$ . Is the operator  $(\mathcal{C}^1([0, 1]), || \cdot ||_{\infty}) \to (\mathcal{C}([0, 1]), || \cdot ||_{\infty})$  $||_{\infty}): f \mapsto f'$  bounded?
- (iv) (Shift operator on  $L^p$ ). Let  $a \in \mathbb{R}$  and  $p \in [1, \infty]$  be given. Consider the mapping  $T_a$  given for a  $f \in L^p(\mathbb{R})$  by prescription

$$(T_a f)(x) = f(x - a)$$
 for a.e.  $x \in \mathbb{R}$ .

226 Clearly  $T_a$  is a linear operator and  $||T_af||_p = ||f||_p$ . Hence,  $T_a: L^p(\mathbb{R}) \to L^p(\mathbb{R})$  is bounded with  $||T_a|| = 1$ . Observe that  $T_a$  is a bijection.

(v) (Shift operators on  $\ell_p$ ). For any  $1 \le p \le \infty$ , define the right shift  $S_R \colon \ell_p \to \ell_p$  and the left shift  $S_L \colon \ell_p \to \ell_p$  by

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$$S_R(x_1, x_2, x_3, \ldots) \coloneqq (0, x_1, x_2, \ldots),$$

231 
$$S_L(x_1, x_2, x_3, \ldots) \coloneqq (x_2, x_3, x_4, \ldots).$$

Verify that these are bounded linear operators, compute their norms, and check whether they are injective or surjective.

(vi) (Multiplication operator). Let  $\Omega \subset \mathbb{R}$  be open and let  $g \in L^{\infty}(\Omega)$  be given. Consider the *multiplication operator*, which, for an  $f \in L^{p}(\Omega), 1 \leq p \leq \infty$ , is given by

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$$(M_g f)(x) = f(x) g(x)$$
 for a.e.  $x \in \mathbb{R}$ 

238 Compute the norm of  $M_g \colon L^p(\Omega) \to L^p(\Omega)$ .

(vii) Consider the indefinite integral operator, for 
$$f \in \mathcal{C}([a, b])$$
,  $a < b$ , given by  

$$Tf(x) = \int_{a}^{x} f(s) \, ds \quad \text{for all } x \in [a, b].$$
Show that  $T: (\mathcal{C}([a, b]), \|\cdot\|_{\infty}) \to \mathcal{C}([a, b]), \|\cdot\|_{\infty})$  is bounded and that  $\|T\| = b - a.$ 
Do you know how can be the range of  $T: L^{1}((a, b)) \to \mathcal{C}([a, b])$  described?
Solution.
(i) We have
$$\|T_A\| = \sup_{\|\|\cdot\|\| \ge 1} \|T_A(x)\|_2 = \sup_{\|\|\cdot\|\| \ge 1} \|Ax\|\|_2 = \|A\|\|_2,$$
the spectral norm of  $A$ . The norm is attained by any dominant right singular vector: If  $A = \sum_{q \in u_1 \vee T} ||x_1|_q| = \sigma_1 = \|A\|_2.$ 
(ii) Suppose that  $p < \infty$ . We estimate
$$\|T_A\|_p^p = \sum_{i=1}^{\infty} |\lambda_i x_i|^p \le \sup_{i \in \mathbb{N}} |\lambda_i|^p \sum_{i=1}^{\infty} |x_i|^p = \|\lambda\|_\infty^p,$$
which implies that
$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|\|_p}{\|x\|\|_p} \le \|\lambda\|_\infty.$$
For
$$x^n = (0, \dots, 0, 1, 0, \dots),$$
it is
$$\|T\| = \sup_{x \neq 0} \frac{\|Tx^n\|_p}{\|x\|\|_p} = \sup_{n \in \mathbb{N}} |\lambda_n| = \|\lambda\|_\infty.$$
Using both inequalities we conclude that  $\|T\| = \|\lambda\|_\infty$  and clearly  $T$  is bounded if and only if  $\lambda \in \ell_\infty$ . We leave the modifications necessary to handle the case  $p = \infty$  for the reader.
(iii) It is
$$\|f_n\|_\infty = 1 \quad \text{and} \quad \|f_n'\|_\infty = n$$
and hence
$$\| \bullet ' \| = \sup_{\|\|I\| \le 1} \|f'\|_\infty \ge \sup_{n \in \mathbb{N}} \|f_n'\|_\infty = \min_{n \in \mathbb{N}} n = \infty,$$

$$\|f_n\|_\infty = 1 \quad \text{and} \quad \|f_n'\|_\infty = n$$
and hence
$$\| \bullet ' \| = \sup_{\|\|I\| \le 1} \|f'\|_\infty \ge \sup_{n \in \mathbb{N}} \|f_n'\|_\infty = \max,$$
(iii) It is
$$\|S_R \|_p^p = \sum_{i=1}^{\infty} |x_i|^p = \|x\|_p^p,$$

270 
$$||S_L x||_p^p = \sum_{i=2}^\infty |x_i|^p \le ||x||_p^p.$$

The inequality becomes an equality if  $x = (0, x_2, x_3, ...)$ . This together shows

|  | that $  S_R   = 1$ and $  S_L   = 1$ . A minor modification shows the same for $p = \infty$ .<br>Given any $p \in [1, \infty]$ , the equation   |
|--|---|
|  | $S_R x = (1, 0, 0, \ldots)$   |
|  | does not have a solution $x \in \ell_p$ and hence $S_R$ is not surjective. On the other hand the equation   |
|  | $S_R x = 0$   |
|  | only has a trivial solution $x = 0$ and hence $S_R$ is injective.<br>Given arbitrary $p \in [1, \infty]$ and $y = (y_1, y_2, \ldots) \in \ell_p$ , the equation   |
|  | $S_L x = y$   |
|  | has a solution, for example, $x = (0, y_1, y_2, y_3, \ldots)$ and hence $S_L$ is surjective.<br>On the other hand, the equation   |
|  | $S_L x = 0$   |
|  | has a non-trivial solution $x = (1, 0, 0,)$ and hence $S_L$ is not injective.   |
| We                                     | eek 4.  |
| Pro<br>erators<br>(i)<br>(ii)<br>(iii) | DBLEM 4.1. On the Banach space $(\mathcal{C}([0,1]), \ \cdot\ _{\infty})$ consider the following op-<br>and decide whether they are linear and bounded:<br>$Tf(x) = f(\cos^2(x)),$<br>$Tf(x) = \cos^2(f(x)),$<br>Tf(x) = -f(0)f'(x) |

(iii) Tf(x)

(iii) Tf(x) = f(0)f'(x), (iv)  $Tf(x) = (x-1)xf(0) + \int_0^x f(s) ds$ , (v) Tf(x) = y(x), where y is the solution of the initial value problem y' + y = fin (0, 1), y(0) = 0.

- Solution.
- (i) T is clearly linear and also bounded. Indeed, for arbitrary  $x \in [0, 1]$ , it is

295 
$$|f(\cos^2 x)| \le \max_{t \in [0,1]} |f(t)| = ||f||_{\infty}$$

$$\begin{array}{ll} \text{Hence } \|Tf\|_{\infty} &= \max_{x \in [0,1]} |f(\cos^2 x)| \leq \|f\|_{\infty}, \text{ which shows that } \|T\| \leq 1. \\ \text{Choosing } f \equiv 1 \text{ shows that } \|T\| = 1. \\ \text{297} & \text{Choosing } f \equiv 1 \text{ shows that } \|T\| = 1. \\ \text{298} & \text{(ii) } T \text{ is clearly non-linear.} \\ \text{300} & \text{(iv) } T \text{ is linear and, for arbitrary } x \in [0,1], \\ \text{301} & |Tf(x)| \leq |f(0)| \, |x-1| \, |x| + \left| \int_0^x f(s) \, \mathrm{d}s \right| \leq \frac{1}{4} |f(0)| + \int_0^1 |f(s)| \, \mathrm{d}s \\ \text{302} & \leq \frac{1}{4} \|f\|_{\infty} + \|f\|_{\infty}. \end{array}$$

Hence  $||T|| \leq \frac{5}{4}$  and T is bounded. 

304 (v) For  $f_1, f_2 \in C([0,1])$ , consider  $y_1, y_2 \in C([0,1])$  such that

$$y'_1 + y_1 = f_1 \qquad \text{in } (0,1), \qquad \qquad y_1(0) = 0, y'_2 + y_2 = f_2 \qquad \text{in } (0,1), \qquad \qquad y_2(0) = 0.$$

307 Due to the linearity of the equations, we have

$$(y_1 + y_2)' + (y_1 + y_2) = (f_1 + f_2)$$
 in  $(0, 1)$ ,  $(y_1 + y_2)(0) = 0$ ,

which shows that  $T(f_1 + f_2) = Tf_1 + Tf_2$ . Proceeding similarly for homogeneity, we get that T is linear.

311 It is readily verified that T has the explicit representation

312 
$$Tf(x) = \int_0^x \exp(t-x) f(t) dt$$

Hence, for any  $x \in [0, 1]$ ,

314 
$$|Tf(x)| \le \int_0^x \exp(t-x) |f(t)| \, \mathrm{d}t \le \int_0^x |f(t)| \, \mathrm{d}t \le ||f||_\infty.$$

315 Hence, T is bounded with  $||T|| \le 1$ .

PROBLEM 4.2 (inequality used in [1, proof of Lemma 2.24]). Let  $f: [0, \infty) \to \mathbb{R}$ be concave such that  $f(0) \ge 0$ . Show that then  $f(a+b) \le f(a) + f(b)$  for all  $a, b \ge 0$ .

318 Solution. By hypotheses, we have, with  $t \in [0, \infty)$  and  $0 \le \lambda \le 1$ , that

319 
$$f(\lambda t) = f(\lambda t + (1 - \lambda)0) \ge \lambda f(t) + (1 - \lambda)f(0) \ge \lambda f(t).$$

320 Hence,

321 322

$$f(a) + f(b) = f\left(\frac{a}{a+b}(a+b)\right) + f\left(\frac{b}{a+b}(a+b)\right)$$
$$\geq \frac{a}{a+b}f(a+b) + \frac{b}{a+b}f(a+b) = f(a+b).$$

323 EXAMPLE 4.3 (examples of Fréchet spaces [1, Examples 2.25, 2.26]).

324 Week 5.

EXAMPLE 5.1 (Schwartz space of rapidly decreasing functions [2]). The Schwartz space (the space of rapidly decreasing functions)

327 
$$\mathcal{S}(\mathbb{R}^n) \coloneqq \left\{ u \in C^{\infty}(\mathbb{R}^n), \, \|x^{\beta} \partial_{\alpha} u\|_{\infty} < \infty \text{ for all multiindices } \alpha, \, \beta \right\}$$

is a Fréchet space (without proof) when equipped with the sequence of seminorms  $\{p_j\}_{j=0}^{\infty}$ ,

330 
$$p_j(u) \coloneqq \sum_{|\alpha|, |\beta| \le j} \|x^\beta \partial_\alpha u\|_{\infty},$$

331 or, for example,  $\{q_j\}_{j=0}^{\infty}$ ,

332 
$$q_j(u) \coloneqq \max_{|\alpha| \le j} \|(1+|x|^2)^j \partial_\alpha u\|_\infty$$

10

305

306

PROBLEM 5.2 (Minkowski functional). Let X be a real normed space and  $B \subset X$ be a non-empty convex open set containing the origin. Let the functional  $p: X \to [0, \infty)$  be defined by

341 
$$p(x) \coloneqq \inf\{\lambda > 0, x \in \lambda B\},$$
 for every  $x \in X$ .

### 342 Show that

(i) there exists M > 0 such that  $p(x) \le M ||x||$  for all  $x \in X$ ; (ii)  $B = \{x \in X, p(x) < 1\}$ ; (iii) p is sublinear, i.e.,

346 
$$p(\alpha x) = \alpha p(x)$$
 for all  $x \in X$  and  $\alpha \ge 0$  and  
347  $p(x+y) \le p(x) + p(y)$  for all  $x, y \in X$ .

### 348 Solution.

(i) By the hypothesis, there exists a ball  $B_r := \{x \in X, \|x\| < r\}$  with certain r > 0 such that  $B_r \subset B$ . Hence

$$p(x) = \inf\{\lambda > 0, \ \frac{x}{\lambda} \in B\} \le \inf\{\lambda > 0, \ \frac{x}{\lambda} \in B_r\} = \frac{\|x\|}{r}$$

(ii) To show " $\subset$ ", suppose that  $x \in B$ . As B is open,  $(1+\delta)x \in B$  for some  $\delta > 0$ small enough. In the other words,  $\frac{x}{\lambda} \in B$  for  $\lambda = \frac{1}{1+\delta}$ , and hence

354 
$$p(x) = \inf\{\lambda > 0, \frac{x}{\lambda} \in B\} \le \inf\left\{\frac{1}{1+\delta}\right\} = \frac{1}{1+\delta} < 1.$$

For the opposite inclusion, suppose that p(x) < 1. By the definition of p, there exists  $0 < \beta < 1$  such that  $\beta \in \{\lambda > 0, x/\lambda \in B\}$ , and hence  $x/\beta \in B$ . As B is convex and contains the origin, we have

$$x = \beta \, \frac{x}{\beta} + (1 - \beta) \, 0 \in B$$

(iii) We leave the task to verify positive homogeneity,  $p(\alpha x) = \alpha p(x)$ , for all  $x \in X$ and  $\alpha \ge 0$ , up to the reader, so it remains to prove the triangle inequality. Suppose that  $x, y \in X$  and fix  $\varepsilon > 0$ . Then for  $\frac{x}{p(x)+\varepsilon}$ , we have

$$p\left(\frac{x}{p(x)+\varepsilon}\right) = \frac{p(x)}{p(x)+\varepsilon} < 1,$$

where the equality follows from the positive homogeneity, and hence, by virtue of (ii),  $\frac{x}{p(x)+\varepsilon} \in B$ . Similarly,  $\frac{y}{p(y)+\varepsilon} \in B$ . By the convexity of B, it follows that, with arbitrary  $0 < \mu < 1$ ,

$$\mu \frac{x}{p(x) + \varepsilon} + (1 - \mu) \frac{y}{p(y) + \varepsilon} \in B.$$

367 Chossing  $\mu \coloneqq \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$  and using (ii) and the positive homogeneity yields  $\begin{pmatrix} x+y \\ p(x+y) \end{pmatrix} = p(x+y)$ 

$$1 > p\left(\frac{x+y}{p(x)+p(y)+2\varepsilon}\right) = \frac{p(x+y)}{p(x)+p(y)+2\varepsilon}.$$

369 As 
$$\varepsilon$$
 was arbitrary, it is  $p(x+y) \le p(x) + p(y)$ .

HOMEWORK 5 (Hahn–Banach separation theorem, weak topology). For a function  $f: X \to \mathbb{R}$ , its *epigraph* is defined as

$$\operatorname{epi} f \coloneqq \{(x, y) \in X \times \mathbb{R}, y \ge f(x)\}.$$

373 LEMMA 5.1. Let X be a convex subset of a real vector space and suppose that 374  $f: X \to \mathbb{R}$  is convex. Then epi f is convex.

375 If X is a normed space, the product  $X \times \mathbb{R}$  is a normed space with, e.g.,  $||(x, y)||_{X \times \mathbb{R}} \coloneqq$ 

376  $||x||_X + |y|$ . Recall we say that a function  $f: X \to \mathbb{R}$  is (norm) lower semicontinuous 377 if  $x_n \to x$  (in norm) implies  $\liminf_{n \to \infty} f(x_n) \ge f(x)$ .

LEMMA 5.2. Suppose that X is a normed space and  $f: X \to \mathbb{R}$  is (norm) lower semicontinuous. Then epi f is (norm) closed.

We say that a subset  $M \subset X$  of a normed space X is (sequentially) weakly closed if every weakly convergent sequence  $\{x_n\}_{n\geq 1} \subset M$  satisfies  $x_n \rightharpoonup x \in M$ . We can immediately see that a weakly closed set is closed. Indeed, suppose that  $\{x_n\} \subset M$ converges in norm to  $x \in X$ . Then  $\{x_n\}$  converges weakly to the same x. As M is weakly closed, it is necessarily  $x \in M$ . The converse holds true for convex sets:

LEMMA 5.3. A subset of a normed space that is closed and convex is weakly closed.

We say that  $f: X \to \mathbb{R}$  is *weakly lower semicontinuous* if the weak convergence  $x_n \rightharpoonup x$ implies  $\liminf_{n \to \infty} f(x_n) \ge f(x)$ .

THEOREM 5.4. Let f be a real-valued functional on a normed space which is lower semicontinuous and convex. Suppose additionally that f is bounded from below. Then f is weakly lower semicontinuous.

391 COROLLARY 5.5. Let V be a normed space. Then the norm  $\|\cdot\|: V \to \mathbb{R}: x \mapsto \|x\|$ 392 is weakly lower semicontinuous, i.e.,

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 $\liminf_{n \to \infty} \|x_n\| \ge \|x\| \qquad \text{whenever } x_n \rightharpoonup x.$ 

Prove the lemmas, the theorem, and the corollary. Lemma 5.3 can be proved by contradiction, invoking the Hahn–Banach (strict) separation theorem. (Recall that any singleton set is compact). The lemmas are all to be proved independently and shall all be used to prove the theorem. It is sufficient to carry out all the proofs for a normed space *over reals* only; the complex case will be treated later in the class.

### 399 Week 6.

400 PROBLEM 6.1 (complex Hahn–Banach theorem).

- 401 (i) Let V be a vector space over  $\mathbb{C}$ . Show that V is a vector space over  $\mathbb{R}$ .
- 402 (ii) Let  $f: V \to \mathbb{C}$  be a linear functional on the complex vector space V. Define 403  $f_1, f_2: V \to \mathbb{R}$  by
- 404  $f_1(x) \coloneqq \operatorname{Re} f(x),$ 405  $f_2(x) \coloneqq \operatorname{Im} f(x).$

$$f_{2}(x) \coloneqq \operatorname{Im} f(x)$$

- Show that  $f_1$  and  $f_2$  are linear functionals on V over  $\mathbb{R}$ , but they are not, in 406 general, linear functionals on V over  $\mathbb{C}$ . 407 (iii) Show that  $f_2(x) = -f_1(ix)$ , and hence  $f(x) = f_1(x) - if_1(ix)$ . 408 (iv) Let X be a complex vector space,  $p: X \to \mathbb{R}$  be a seminorm, and let  $V \subset X$ 409be a subspace of X. Suppose that  $f: V \to \mathbb{C}$  is linear such that  $|f(x)| \leq p(x)$ 410 on V. Apply the real version of Hahn–Banach theorem to construct a linear 411  $F_1: X \to \mathbb{R}$ , an extension of  $f_1: V \to \mathbb{R}$ , such that  $|F_1| \leq p$  on X. 412 (v) From  $F_1$  construct a linear  $F: X \to \mathbb{C}$ , an extension of  $f: V \to \mathbb{C}$ , and show 413that  $|F| \leq p$  on X. 414415 Solution. (ii) For arbitrary  $x, y \in V$ , we have  $f_1(x+y) = \operatorname{Re} f(x+y) = \operatorname{Re} f(x) + \operatorname{Re} f(y) =$ 416  $f_1(x) + f_1(y)$ . As of homogeneity, we have  $f_1(\lambda x) = \operatorname{Re} f(\lambda x) = \operatorname{Re}(\lambda f(x))$  for 417 any  $\lambda \in \mathbb{C}$ . If  $\lambda$  is real, then the last expression equals  $\lambda f_1(x)$ , which shows 418 that  $f_1$  is linear on V over  $\mathbb{R}$ . On the other hand, homogeneity  $f_1(\lambda x) =$ 419 $\lambda f_1(x)$  is clearly violated if, for example,  $\lambda = i$  and  $f_1(x) \neq 0$ . Indeed, the 420 left-hand side is real and the right-hand side is imaginary. 421 (iii) Indeed, for any  $x \in V$ , we have  $f_1(ix) = \operatorname{Re} f(ix) = \operatorname{Re}(if(x)) = -f_2(x)$ . 422(iv) Linear functional  $f_1: V \to \mathbb{R}$  is dominated by p on V. Indeed,  $|f_1(x)| =$ 423  $|\operatorname{Re} f(x)| \leq |f(x)| \leq p(x)$ . By the real Hahn–Banach theorem, there exists 424  $F_1: X \to \mathbb{R}$ , a linear functional on X over  $\mathbb{R}$ , such that  $F_1 = f_1$  on V and 425  $F_1 \leq p$  on X. As p is a seminorm (recall that a sublinear function which is 426additionally absolute homogeneous is a seminorm), it is  $-F_1(x) = F_1(-x) \leq$ 427 p(-x) = p(x), which shows, together with  $F_1(x) \le p(x)$ , that  $|F_1| \le p$  on X. 428 (v) For an arbitrary  $x \in X$ , let  $F(x) \coloneqq F_1(x) - iF_1(ix)$ . It is readily verified, 429directly from the definition, that F is a linear functional on X over  $\mathbb{C}$ . It is 430also an extension of f. Indeed, for  $x \in V$ , it is  $F(x) = F_1(x) - iF_1(ix) =$ 431 $f_1(x) - i f_1(ix) = f(x)$ . It remains to verify that |F| is dominated by p. 432 Let  $x \in X$  be arbitrary and fixed. There exists  $t \in \mathbb{R}$  such that |F(x)| =433 $e^{it}F(x) = F(e^{it}x) = F_1(e^{it}x) - iF_1(ie^{it}x)$ . The left-hand side is real and  $F_1$ 434 is real-valued so it must be  $|F(x)| = F_1(e^{it}x) \le p(e^{it}x) = |e^{it}|p(x) = p(x).$ 435Thus we have proved the complex version of the Hahn–Banach theorem: 436 COROLLARY. Let X be a complex vector space and  $V \subset X$  be its subspace. Let 437 $f: V \to \mathbb{C}$  be linear,  $p: X \to \mathbb{R}$  be a seminorm, and  $|f| \leq p$  on V. Then there exists 438 a linear  $F: X \to \mathbb{C}$  such that F = f on V and  $|F| \leq p$  on X. 439
- 440 PROBLEM 6.2 (Mazur's lemma). Let X be a real vector space and  $M \subset X$  be an 441 arbitrary set. We define the *convex hull* of M as
- 442 conv  $M \coloneqq \{x \in X, x \text{ is a finite convex combination of elements of } M\}$

443 
$$= \begin{cases} x \in X, \text{ there exists } m \in \mathbb{N}, \text{ positive numbers } \lambda_1, \dots, \lambda_m \text{ with} \\ \sum_{j=1}^m \lambda_j = 1, \text{ and vectors } x_1, \dots, x_m \in M \text{ such that } x = \sum_{j=1}^m \lambda_j x_j \end{cases}$$

- (i) Show that  $M \subset \operatorname{conv} M$  and that  $\operatorname{conv} M$  is convex.
- (ii) Use Lemma 5.3 to prove the following result:
- 446 THEOREM (Mazur's lemma). Let X be a real normed space and suppose that 447  $\{x_i\}_{i=1}^{\infty} \subset X$  converges weakly to some  $x \in X$ . Then  $x \in \operatorname{conv}\{x_i\}_{i=1}^{\infty}$ .
- 448 (iii) Show that this statement is equivalently formulated as follows:

449 THEOREM (Mazur's lemma). Let X be a real normed space and suppose that 450  $\{x_j\}_{j=1}^{\infty} \subset X$  converges weakly to some  $x \in X$ . Then there exists a sequence 451 of finite convex combinations of  $\{x_j\}_{j=1}^{\infty}$  which converges strongly to x. Pre-452 cisely, there exists a sequence of integers  $\{m_j\}_{j=1}^{\infty}$  and numbers  $0 \le \lambda_{ji} \le 1$ , 453  $j = 1, 2, ..., i = 1, 2, ..., m_j$ , with  $\sum_{i=1}^{m_j} \lambda_{ji} = 1$  for every  $j \in \mathbb{N}$ , such that

$$\sum_{i=1}^{m_j} \lambda_{ji} x_i \to x \qquad strongly \ as \ j \to \infty$$

455 Week 7.

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456 PROBLEM 7.1 (on separability).

(i) Show that every subset of a separable metric space is separable.

- (ii) Show that  $\ell_p$  is separable for every  $1 \le p < \infty$  and that  $\ell_{\infty}$  is not separable.
- (iii) Let  $\Omega \subset \mathbb{R}^d$  be open. Show that  $L^p(\Omega)$  is separable for every  $1 \leq p < \infty$  and that, provided  $\Omega$  is not empty,  $L^{\infty}(\Omega)$  is not separable.

461 PROBLEM 7.2 (Baire property). Let X be a topological space. Show that the 462 following properties are equivalent.

463 (i) Every countable union of closed sets with empty interior has empty interior.
464 (ii) Every countable intesection of dense open sets is dense.

We say that a set is a nowhere dense subset of X if its closure has empty interior. We say that a subset of X is a meager subset of X, meager in X, or of the first category in X if it is a countable union of nowhere dense subsets of X. A subset of X which is not meager in X is called a nonmeager subset of X, nonmeager in X, or of the second category in X. Then the Baire property (i), (ii) is equivalently expressed as follows.

470 (iii) Every meager subset of X has empty interior.

(iv) Every nonempty open subset of X is nonmeager in X.

472 PROBLEM 7.3 (Everywhere-defined unbounded operator on a Banach space). 473 Let X be an infinite dimensional vector space. We say that a set  $M = \{v_i\}_{i \in I}$  is 474 *linearly independent* if for every *finite* index set  $J \subset I$ , the equation  $\sum_{j \in J} c_j v_j = 0$ 475 implies that  $c_j = 0$  for all  $j \in J$ . We say that a set  $B \subset X$  is a *Hamel basis* of X 476 if B is linearly independent and every element of X can be written as a *finite* linear 477 combination of elements of B.

(i) Let a linearly independent sequence  $\{b_i\}_{i=0}^{\infty} \subset X$  be given. Show, using Zorn's lemma, that there exists a Hamel basis *B* containing  $\{b_i\}_{i=0}^{\infty}$  as its subset.



- (ii) Let X be a Banach space. Recall that, by the Baire category theorem,  $\{b_i\}_{i=0}^{\infty}$ alone cannot be a Hamel basis of X. In the other words, the dimension of X is uncountably infinite.
- (iii) Now assume w.l.o.g. that  $||b_i|| = 1$  for i = 1, 2, ... and consider the function  $F: B \to \mathbb{R}$  given as  $F(b_i) = i$  for i = 1, 2, ... and F(b) = 0 for  $b \in B \setminus \{b_i\}_{i=1}^{\infty}$ . Show that F is uniquely extended to a *linear* functional  $F: X \to \mathbb{R}$ . Show that F is unbounded.

488 Homework 6.

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- (i) Show that the unit ball in  $L^2((0,1))$ , i.e., the set  $\{f \in L^2((0,1)), \|f\|_2 < 1\}$ , is a nowhere dense subset of  $L^1((0,1))$ .
- (ii) Building on (i), decide whether  $L^2((0,1))$  is a meager or nonmeager subset of  $L^1((0,1))$ .

Homework 7.

- (i) Recall that for every vector space a Hamel basis exists by Zorn's lemma;
   (f. Problem 7.3 (i). Use the Baire category theorem to show that for every
   (i) infinite-dimensional Banach space its Hamel basis is uncountable.
- (ii) Consider  $\mathcal{P}$ , the space of polynomials in one variable of arbitrary degree with real coefficients. Show the following statement: There does not exist a functional  $\|\cdot\|: \mathcal{P} \to \mathbb{R}$  such that  $(\mathcal{P}, \|\cdot\|)$  is a Banach space.

HOMEWORK 8. Let U be a Banach space and let  $T: U \to \ell_{\infty}$  be a linear operator defined on whole U, i.e., such that  $Tx \in \ell_{\infty}$  for every  $x \in U$ . Consider its components  $T_j: U \to \mathbb{R}$  given by  $T_j(x) = (Tx)_j$  for  $x \in U$  and  $j \in \mathbb{N}$ . Prove that T is bounded if and only if  $T_j, j \in \mathbb{N}$ , are all bounded.

504 Homework 9.

- (i) Show that there exists a bounded linear functional F on  $\ell_{\infty}$  such that  $F(x) = \lim_{k \to \infty} x_k$  whenever x is a convergent sequence.
- 507 (ii) Show that there exists a bounded linear functional  $F \in L^{\infty}(\mathbb{R})^*$  such that 508  $F(f) = \operatorname{ess} \lim_{x \to 0} f(x)$  whenever the limit exists.
- 509 (iii) Show that (ii) fails when  $L^{\infty}(\mathbb{R})$  is replaced by  $L^{1}(\mathbb{R})$ . To do this, find 510 a bounded sequence  $\{f_n\}_{n=1}^{\infty} \subset L^{1}(\mathbb{R})$  with  $F(f_n) \to \infty$  as  $n \to \infty$ .

### 511 Week 8.

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512 PROBLEM 8.1 (dual of  $L^p$ ). Let  $\Omega \subset \mathbb{R}^d$  be open. Let  $p \in (1, \infty)$  and 1/p + 1/p' =513 1. In the sequel we will use the notation  $L^p \coloneqq L^p(\Omega)$  and  $(L^p)^* \coloneqq (L^p(\Omega))^*$  for any

514  $1 . Consider the mapping <math>T: L^{p'} \to (L^p)^*$  given by

$$\langle Tu, f \rangle = \int_{\Omega} u f \, \mathrm{d}x, \qquad f \in L^p.$$

516 (i) Show that T is linear.

- (ii) Show that T is isometry; precisely  $||Tu||_{(L^p)^*} = ||u||_{p'}$  for every  $u \in L^{p'}$ .
- 518 (iii) Show that  $T(L^{p'})$ , the range of T, is closed in  $(L^p)^*$ .
- (iv) Show that  $T(L^{p'})$  is dense in  $(L^p)^*$ . Use reflexivity of  $L^{p'}$  and the following proposition.
- 521 LEMMA 8.1. Let V be a normed space and  $M \subset V$  be its subspace. Then 522  $\overline{M} = V$  if and only if

523 
$$\{F \in V^*, F = 0 \text{ on } M\} = \{F \in V^*, F = 0 \text{ on } V\}.$$

524 (v) Conclude that, for  $1 , <math>(L^p)^*$  is isometrically isomorphic to  $L^{p'}$ 525 (through T).

526 Proof of Lemma 8.1. Suppose that M is dense. If F = 0 on M and  $\{x_k\} \subset M$  is 527 such that  $x_k \to x \in \overline{M} = V$ , then  $0 = F(x_k) \to F(x)$  by virtue of continuity of F. 528 As  $x \in V$  was arbitrary, this shows that F = 0 on V.

For the opposite implication, suppose that  $\overline{M}$  is a proper subspace of V. We use the following proposition from the class:

THEOREM 8.2 (a consequence of the Hahn–Banach theorem). Let M be a closed proper subspace of a normed space V and let  $x \in V \setminus M$  be given. Then there exists  $F \in V^*$  such that F = 0 on M, ||F|| = 1, and F(x) = dist(x, M) > 0.

- Thus, there exists a non-zero F that vanishes on  $\overline{M}$ , and, in particular, on M. Solution.
- 536 (ii) Given an arbitrary  $u \in L^{p'}$ , we obtain, using the definition of T and the 537 Hölder inequality,

538 
$$||Tu||_{(L^p)^*} = \sup_{f \in L^p} \frac{\langle Tu, f \rangle}{||f||_p} = \sup_{f \in L^p} \frac{\int uf}{||f||_p} \le ||u||_{p'}.$$

539 On the other hand, the function  $f_u := |u|^{p'-2}u$  belongs to  $L^p$ , which is easily 540 verfied by checking that (p'-1)p = p', and it is  $||f_u||_p = ||u||_{p'}^{p'-1}$ . Hence

541 
$$\|Tu\|_{(L^p)^*} = \sup_{f \in L^p} \frac{\int u f}{\|f\|_p} \ge \frac{\int u f_u}{\|f_u\|_p} = \frac{\|u\|_{p'}^{p'}}{\|u\|_{p'}^{p'-1}} = \|u\|_{p'}.$$

542 Altogether we have that  $||u||_{p'} \leq ||Tu||_{(L^p)^*} \leq ||u||_{p'}$ , which shows that the 543 inequality is actually an equality.

- (iv) Denote  $E := T(L^{p'})$ . To show that  $\overline{E} = (L^p)^*$ , it is sufficient (and necessary) 544by Lemma 8.1 to show that: if an arbitrary  $h \in (L^p)^{**}$  vanishes on E then 545h = 0. Suppose that  $h \in (L^p)^{**}$  vanishes on E, i.e.,  $\langle h, Tu \rangle_{(L^p)^{**}, (L^p)^*} = 0$  for 546 every  $u \in L^{p'}$ . As  $L^p$  is reflexive for  $1 , there exists <math>h \in L^p$  (denoted the same as  $h \in (L^p)^{**}$  such that  $\langle h, F \rangle_{(L^p)^{**}, (L^p)^*} = \langle F, h \rangle_{(L^p)^*, L^p}$  for every  $F \in (L^p)^*$ . For  $F \coloneqq Tu$ , we have  $0 = \langle h, Tu \rangle_{(L^p)^{**}, (L^p)^*} = \langle Tu, h \rangle_{(L^p)^*, L^p} = \langle Tu, h \rangle_{(L^p)^*, L^p}$ 548549 $\int_{\Omega} u h$  for every  $u \in L^{p'}$ . The choice  $u \coloneqq |h|^{p-2}h$  is an admissible test function from  $L^{p'}$ , which is easily verified by checking that (p-1)p' = p. 551Thus  $0 = \int_{\Omega} |h|^p = ||h||_p^p$ , hence  $h \in L^p$  is the zero function, and, by the 552isometry of the canonical embedding,  $h \in (L^p)^{**}$  is the zero functional.
- (v) Isometry of T immediatelly implies that T is injective. As  $E := T(L^{p'})$ , the range of T, is closed and dense, it is  $E = \overline{E} = (L^p)^*$ . Hence T is surjective.

## 556 Week 9.

557 PROBLEM 9.1. On the Banach space C([0, 1]) consider the following operators and 558 decide whether they are compact linear operators.

559 (i)  $Tf(x) = f(\cos^2(x)),$ 

560 (ii)  $Tf(x) = \cos^2(f(x)),$ 

561 (iii) Tf(x) = f(0)f'(x),

- 562 (iv)  $Tf(x) = (x-1)xf(0) + \int_0^x f(s) \, \mathrm{d}s$ ,
- 563 (v) Tf(x) = y(x), where y is the solution of the initial value problem y' + y = f564 in (0, 1), y(0) = 0.

FROBLEM 9.2 (compact embedding of Hölder spaces). Let  $\Omega \subset \mathbb{R}^d$  be open and bounded and let  $0 < \alpha < \beta \leq 1$  be given. Show that  $C^{0,\beta}(\Omega)$  is compactly embedded in  $C^{0,\alpha}(\Omega)$ , i.e., show that the identity mapping from  $C^{0,\beta}(\Omega)$  to  $C^{0,\alpha}(\Omega)$  is compact. Use the Arzelà–Ascoli theorem.

FOR PROBLEM 9.3 (compactness of integral operator). Let  $K: [a,b] \times [a,b] \to \mathbb{R}$  be a continuous function. Show that the integral operator  $T: \mathcal{C}([a,b]) \to \mathcal{C}([a,b])$  given by

572 (9.1) 
$$(Tf)(x) = \int_{a}^{b} K(x,y) f(y) \, \mathrm{d}y$$

573 is compact. Use the Arzelà–Ascoli theorem.

For  $f \in \mathcal{C}([-1,1])$  consider the following boundary value problem:

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$$-u'' = f \quad \text{in } (-1,1), \qquad u(-1) = u(1) = 0$$

576 Show that the solution to this problem is unique and that it is represented by the 577 formula

578 
$$u(x) = \int_{-1}^{x} \frac{(1+y)(1-x)}{2} f(y) \, \mathrm{d}y + \int_{x}^{1} \frac{(1-y)(1+x)}{2} f(y) \, \mathrm{d}y.$$

Show that the solution operator  $f \mapsto u$  can be written in the form (9.1) with certain Kand hence it is compact.

581 Solution of Homework 8. The right implication is the easy one. We will prove 582 the left one. Suppose that  $T_j$  are all bounded and  $Tx \in \ell_{\infty}$  for all  $x \in U$ . Let  $x \in U$ 583 be arbitrary and fixed. Then

584 
$$\infty > \|Tx\|_{\infty} = \sup_{j \in \mathbb{N}} |(Tx)_j| = \sup_{j \in \mathbb{N}} |T_j x|$$

As  $\{T_j\}_{j\in\mathbb{N}}$  are all bounded operators from U to  $\mathbb{R}$  and  $\{T_jx\}_{j\in\mathbb{N}}$  is a bounded sequence in  $\mathbb{R}$  for every  $x \in U$ , the uniform boundedness principle yields that  $\{T_j\}_{j\in\mathbb{N}}$  is a bounded sequence of operators. Hence

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$$\infty > \sup_{j \in \mathbb{N}} ||T_j|| = \sup_{j \in \mathbb{N}} \sup_{\|x\|_U = 1} |T_j x| = \sup_{\|x\|_U = 1} \sup_{j \in \mathbb{N}} \sup_{x\|_U = 1} ||T_j x|| = \sup_{\|x\|_U = 1} ||Tx\|_{\infty} = ||T||. \square$$

### 589 Week 10.

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<sup>590</sup> PROBLEM 10.1 (compactness of integral operator). Let  $\tau_h$  denote the shift oper-<sup>591</sup> ator, i.e., given a function  $f \colon \mathbb{R}^d \to X$  and  $h \in \mathbb{R}^d$ , the shifted function  $\tau_h f \colon \mathbb{R}^d \to X$ <sup>592</sup> is given by

$$(\tau_h f)(x) \coloneqq f(x+h), \qquad x \in \mathbb{R}^d.$$

THEOREM (Kolmogorov–M. Riesz–Fréchet). Let  $q \in [1, \infty)$  and  $d \in \mathbb{N}$  be given and let  $\{u_j\}_{j=1}^{\infty}$  be a bounded sequence in  $L^q(\mathbb{R}^d)$ . Suppose that

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$$\lim_{h \to 0} \|\tau_h u_j - u_j\|_q = 0 \qquad uniformly \ in \ j = 1, 2, \dots$$

597 *i.e.*, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\|\tau_h u_j - u_j\|_q < \varepsilon$  for all j = 1, 2, ...,598 and all  $h \in \mathbb{R}^d$  with  $|h| < \delta$ .

599 Let  $\Omega \subset \mathbb{R}^d$  be an arbitrary measurable set with finite measure. Then there exists 600 a subsequence  $\{u_{j_k}\}_{k=1}^{\infty}$  such that  $\{u_{j_k}|_{\Omega}\}_{k=1}^{\infty}$  converges in  $L^q(\Omega)$ .

Let q > 1 be given and  $\Omega \subset \mathbb{R}^d$  be open and bounded. Let  $K: \Omega \times \Omega \to \mathbb{R}$  be 601 uniformly continuous. Show that the integral operator  $T: L^q(a, b) \to L^q(a, b)$  given 602603 by

604 
$$(Tf)(x) = \int_{\Omega} K(x,y) f(y) \, \mathrm{d}y$$

is compact. Use the Kolmogorov criterion. 605

PROBLEM 10.2. Let  $f \in \mathcal{C}([0,1]), q > 1$ , and  $n \in \mathbb{N}$  be given. We consider the 606 approximation problem of finding  $p_* \in \mathcal{P}_n$ , the space of polynomials of degree at 607 most n, that would be the closest to f in the  $L^{q}(0,1)$  norm. Denote 608

$$M \coloneqq \inf_{p \in \mathcal{P}_n} \|f - p\|_q.$$

- (i) Let q = 2. Use the projection theorem in Hilbert spaces to show that M =610  $||f - p_*||_2$  for some  $p_* \in \mathcal{P}_n$ , that  $p_*$  is uniquely given, and that the mapping 611  $f \mapsto p_*$  is linear. 612 (ii) Let q > 1 be arbitrary. Show that there exists a unique  $p_* \in \mathcal{P}_n$  such that 613
- 614  $M = ||f - p_*||_q$  and that the mapping  $f \mapsto p_*$  is nonlinear unless q = 2.
- (ii) We will present the direct method in the calculus of variations. Solution. 615 Step 1 (show that  $M > -\infty$ ). Clearly it is  $M \ge 0$ . One also gets that  $M < \infty$ 616 as for the zero polynomial one gets  $M \leq ||f||_q < \infty$ . 617
  - Step 2 (take a minimizing sequence). This is trivial, from the definition of infimum:  $M = \lim_{k \to \infty} ||f - p_k||_q$  for some sequence  $\{p_k\}_{k=1}^{\infty} \subset \mathcal{P}_n$ .
- Step 3 (establish a limit). If we show that  $\{p_k\}_{k=1}^{\infty}$  is bounded in  $L^q(0,1)$ , 620 the Banach–Alaoglu theorem and reflexivity of  $L^q$  assure that there is 621 a weakly convergent subsequence, i.e.,  $p_{k_j} \rightharpoonup p_*$  weakly in  $L^q$ . So it re-622 mains to show the boundedness:  $||p_k||_q \leq ||p_k - f||_q + ||f||_q \rightarrow M + ||f||_q <$ 623 624  $\infty$ , i.e., the sequence  $||p_k||_q$  is dominated by a convergent sequence.
  - Step 4 (show inclusion of the limit in the trial space).  $\mathcal{P}_n$  is a finite-dimensional subspace of  $L^{q}(0,1)$ , hence closed and in turn, by Lemma 5.3, weakly closed. So the weak convergence  $p_{k_i} \rightharpoonup p_*$  implies  $p_* \in \mathcal{P}_n$ .
- Step 5 (pass to the limit in the functional). Our functional is  $p \mapsto F(p) := ||f f(p)| = ||f|$ 628  $p||_q$ . In this step one wants to show that  $F(p_*) = M$ . As  $F(p_{k_i}) \to M$  by 629construction, this step amounts to showing that  $F(p_{k_i}) \to F(p_*)$ . So far 630 we have established that  $p_{k_i}$  is weakly convergent. It is left as an exercise 631 to show that  $F: L^q(0,1) \to \mathbb{R}$  given above is continuous and convex. We 632 can use Theorem 5.4 to deduce that F is weakly lower semicontinuous. 633 Hence the weak convergence implies that  $\liminf_{j\to\infty} F(p_{k_j}) \geq F(p_*)$ . In 634 the other words, 635

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$$M = \lim_{k \to \infty} \|f - p_k\|_q = \liminf_{j \to \infty} \|f - p_{k_j}\|_q \ge \|f - p_*\|_q \ge M.$$

Both the left-hand side and the right-hand side are M, so we conclude 637 that  $M = ||f - p_*||_q$ . 638

639 Step 6 (uniqueness). Suppose that 
$$||f - p_1||_q = ||f - p_2||_q = M$$
 for distinct  $p_1$ ,  
640  $p_2 \in \mathcal{P}_n$ . Then, for arbitrary fixed  $\lambda \in (0, 1)$ ,

641 
$$\|f - \lambda p_1 - (1 - \lambda)p_2\|_q = \|\lambda(f - p_1) + (1 - \lambda)(f - p_2)\|_q < \lambda \|f - p_1\|_q + (1 - \lambda)\|f - p_2\|_q = M,$$

647 PROBLEM 10.3. Show that every Hilbert space is reflexive. Use the Riesz repre-648 sentation theorem.

649 PROBLEM 10.4. Prove the following Hilbert space analog of Lemma 8.1. Notice 650 that the Hahn–Banach theorem is not needed.

651 THEOREM. Let H be a Hilbert space and  $M \subset H$  be its subspace. Then  $\overline{M} = H$ 652 if and only if

653 
$$\{x \in H, (x, y) = 0 \text{ for all } y \in M\} = \{0\}$$

654 PROBLEM 10.5 (orthogonal complement). Let V be a real or complex inner-655 product space and suppose that M is a non-empty subset of V. We define the 656 orthogonal complement of M as

657 
$$M^{\perp} \coloneqq \{x \in V, (x, y) = 0 \text{ for all } y \in M\}$$

THEOREM. Let V be an inner product space and let M be its non-empty subset. Then  $M^{\perp}$  is a closed subspace of V. Furthermore,  $(\overline{M})^{\perp} = M^{\perp}$  and  $M \cap M^{\perp} = \{0\}$ if  $0 \in M$  and  $M \cap M^{\perp} = \emptyset$  if  $0 \notin M$ .

661 COROLLARY (direct sum theorem). Let H be a real or complex Hilbert space and 662 let M be a closed subspace of H. Then  $H = M \oplus M^{\perp}$ .

663 PROBLEM 10.6 (adjoint).

664 THEOREM. Let X, Y be complex Hilbert spaces and let  $A \in \mathcal{L}(X, Y)$  be given. 665 There exists a uniquely given  $A^* \in \mathcal{L}(Y, X)$ , the adjoint of A, that satisfies

$$(Ax, y)_Y = (x, A^*y)_X$$
 for all  $x \in X, y \in Y$ 

667 Furthermore,

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668 $(\operatorname{Im} A)^{\perp} = \operatorname{Ker} A^*,$  $Y = \operatorname{Ker} A^* \oplus \overline{\operatorname{Im} A},$ 669 $(\operatorname{Im} A^*)^{\perp} = \operatorname{Ker} A,$  $X = \operatorname{Ker} A \oplus \overline{\operatorname{Im} A^*}.$ 

670 HOMEWORK 10. Let q > 1,  $f \in L^q(0,1)$ , and  $n \in \mathbb{N}$  be given. Consider functional 671  $F : \mathcal{P}_n \to [0,\infty)$  given by

$$F(p) \coloneqq \frac{1}{q} \|f - p\|_q^q$$

(i) Compute the Gateaux derivative of F.

(ii) Formulate the necessary condition for  $p_* = \operatorname{argmin}_{p \in \mathcal{P}_n} F(p)$ .

- (iii) Show that the necessary condition is sufficient.
- (iv) Show that the mapping  $P: L^q(0,1) \to \mathcal{P}_n: f \mapsto p_*$  is a projection onto  $\mathcal{P}_n$ , i.e., it is an idempotent mapping  $(P^2 = P)$  and the range of P is  $\mathcal{P}_n$ .
- 678 (v) Show that P is nonlinear unless q = 2.
- 679 HOMEWORK 11. Solve Problem 9.1 (i), (v).
- 680 HOMEWORK 12. Let  $H := L^2(0, \pi)$  and  $f_n(x) := \sin^2 nx$ .

- 681 (i) Given  $0 \le a \le b \le \pi$ , compute  $\int_0^{\pi} f_n(x) \Xi_{(a,b)}(x) dx = \int_a^b f_n$ , where  $\Xi_{(a,b)}$  is 682 the characteristic function of interval (a, b). Verify that the integral tends to 683  $\frac{b-a}{2}$  as  $n \to \infty$ .
  - (ii) Show that  $\int_0^{\pi} f_n(x)\varphi(x) \, \mathrm{d}x \to \int_0^{\pi} \frac{1}{2}\varphi(x) \, \mathrm{d}x$  for every step function  $\varphi \colon [0,\pi] \to \mathbb{R}$ .
- 686 (iii) Recall that for an arbitrary  $\varphi \in L^2(0,\pi)$  and  $\varepsilon > 0$ , there exists a step 687 function  $\varphi_{\varepsilon}$  such that  $\|\varphi - \varphi_{\varepsilon}\|_2 < \varepsilon$ . Consider the identity
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$$\left(f_n(x) - \frac{1}{2}\right)\varphi(x) \,\mathrm{d}x$$

$$= \int_0^\pi \left(f_n(x) - \frac{1}{2}\right)\left(\varphi(x) - \varphi_\varepsilon(x)\right) \,\mathrm{d}x + \int_0^\pi \left(f_n(x) - \frac{1}{2}\right)\varphi_\varepsilon(x) \,\mathrm{d}x$$

690 and the facts shown above to prove that  $f_n \rightharpoonup \frac{1}{2}$  weakly in  $L^2(0, \pi)$ .

(iv) Show that  $f_n$  does not have a strong limit (in  $L^2(0,\pi)$ ).

(v) Consider linear operator  $T: L^2(0,\pi) \to L^2(0,\pi)$  given by  $Tf(x) = \int_0^x f(y) \, dy$ . This is a bounded linear operator. Indeed,

694  $||Tf||_{2}^{2} = \int_{0}^{\pi} \left| \int_{0}^{x} f(y) \, \mathrm{d}y \right|^{2} \mathrm{d}x$ 695  $< \int_{0}^{\pi} \left( \int_{0}^{x} |f(y)| \, \mathrm{d}y \right)^{2} \mathrm{d}x$ 

 $\int_{0}$ 

$$\leq \int_0^{\pi} \left( \int_0^x |f(y)| \, \mathrm{d}y \right)^2 \mathrm{d}x \leq \int_0^{\pi} \left( \int_0^{\pi} |f(y)| \, \mathrm{d}y \right)^2 \mathrm{d}x \leq \pi^2 ||f||_2^2,$$

where the last inequality follows from the Hölder inequality. Show that T is compact using Kolmogorov's criterion.

(vi) Compute  $Tf_n$  and  $T\frac{1}{2}$ , show that  $||Tf_n - T\frac{1}{2}||_{\infty} \to 0$ , and conclude that  $Tf_n \to T\frac{1}{2}$  strongly in  $L^2(0,\pi)$ .

TOO HOMEWORK 13. Let Y be the subset of  $\ell_2$  given by

$$Y \coloneqq \left\{ x = \{ x_i \}_{i=1}^{\infty} \in \ell_2, \, x_{2k-1} = x_{2k} \text{ for all integers } k \ge 1 \right\}$$

- (i) Show that Y is a closed subspace of  $\ell_2$ .
- (ii) Identify the orthogonal complement of Y in  $\ell_2$ .
- (iii) Identify the  $\ell_2$ -orthogonal projection onto Y.
- 705 HOMEWORK 14.

THEOREM 10.1 (Hahn-Banach theorem in Hilbert spaces). Let H be a Hilbert space,  $Y \subset H$  be a subspace, and  $f \in Y^*$  be an arbitrary linear bounded functional on Y. Then there exists a bounded linear functional  $F \in H^*$  such that F = f on Yand ||F|| = ||f||. Besides, such F is unique.

For the proof of this theorem, carry out the following steps. Notice that Zorn's lemma (axiom of choice) has not been invoked, in contrast to the Hahn–Banach theorem in nonseparable Banach spaces.

(i) Show that there exists a unique continuous extension of f from Y to  $\overline{Y}$ . If  $y \in \overline{Y}$ , then there exists  $\{y_n\} \subset Y$  such that  $y_n \to y$  in norm. Define  $\hat{f}(y) \coloneqq \lim_{n \to \infty} f(y_n)$ . Show that this is a correct definition (i.e,  $y \mapsto \hat{f}(y)$ is a function), that  $\hat{f}$  is linear, and that  $\hat{f}$  is bounded. Show that such  $\hat{f}$  is uniquely given.

(ii) Now for an arbitrary 
$$x \in H$$
, consider its decomposition  $x = Px + (I - P)x$ ,  
where P is the orthogonal projection onto  $\overline{Y}$ , cf. the direct sum theorem. Set

720  $F(x) \coloneqq \hat{f}(Px)$  for all  $x \in H$ , verify that F is a linear extension of f, and 721 compute the norm of F.

(iii) It remains to show uniqueness of F with such properties. This is postponed to the exercise session or is considered a bonus task.

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