On Random Simplex Picking

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Short abstract.

Selected values of odd random simplex volumetric moments are derived in an exact form in various bodies in dimensions three, four, five and six. In three dimensions, the well known Efron's formula is also used. As it turned out, the problem is solvable in higher dimensions too using nothing more than Blashke-Petkanchin formula in Cartesian parametrisation in the form of the Canonical Section Integral.

Long abstract.

Let K_d be a compact and convex body in $\mathbb{R}^d$ with $\dim K_d = d$. One family of such bodies are the d-simplex T_d, d-cube C_d or d-orthoplex O_d (the dual of C_d). Let $\mathbb{X} = (\mathbf{X}_0, \mathbf{X}_1, \dots \mathbf{X}_n)$ be a sample of $(n+1)$ random points $\mathbf{X}_j$, $j = 0, \dots, n$ with $n \geq d$ selected uniformly and independently from the interior of K_d and let $H_n(K_d) = \operatorname{conv}(\mathbb{X}) = \operatorname{conv}(\mathbf{X}_0, \dots, \mathbf{X}_n)$ be the convex hull of those points. We define $\Delta_n = \operatorname{vol}_d H_n(K_d)$ and its corresponding *metric moments*

$$v_n^{(k)}(K_d) = \frac{\mathbb{E}\,\Delta_n^k}{(\operatorname{vol}_d K_d)^k} \tag{1}$$

the normalisation factor in the denominator ensures they stay affinely invariant, that is with respect to affine transformations of K_d. When $n = d$, we refer to $v_d^{(k)}(K_d)$ as the *volumetric moments* in K_d. For even k, (even) volumetric moments are trivial to obtain. For a d-ball B_d, its metric moments $v_n^{(k)}(B_d)$ are known for any n, k and d due to Miles [4], although the special case $v_3^{(1)}(B_3) = 9/715$ was already obtained one hundred years ago by Hostinský [3]. Later, Buchta and Reitzner [2] found $v_3^{(1)}(T_3) = \frac{13}{720} - \frac{\pi^2}{15015}$, subsequently followed by Zinani's [5] $v_3^{(1)}(C_3) = \frac{3977}{216000} - \frac{\pi^2}{2160}$. No other values of odd volumetric moments in three dimensions were known. From December 2020 onwards, we deduced $v_3^{(1)}(O_3) = \frac{19297\pi^2}{3843840} - \frac{6619}{184320}$ and also $v_3^{(1)}(K_3)$ for K_3 being a tetrahedron bipyramid, square pyramid, triangular prism, cuboctahedron, truncated tetrahedron and rhombic dodecahedron (revealed in Salzburg conference in September 2023, see [1]). Later, we also derived some higher odd moments, namely $v_3^{(k)}(P_3)$ for $P_3 = T_3, C_3, O_3$ with $k = 3$ and $k = 5$. For $d \geq 4$, no odd volumetric moments were known for any polytope. This changed in March 2024, most notably, we found

$$\begin{aligned}
v_4^{(1)}(T_4) &= \tfrac{97}{27000} - \tfrac{2173\pi^2}{52026975} \approx 0.0031803708487,\\
v_4^{(3)}(T_4) &= \tfrac{1955399}{3403417500000} + \tfrac{63065881\pi^2}{396699961407750000} \approx 5.9023 \cdot 10^{-7},\\
v_4^{(5)}(T_4) &= \tfrac{12443146181}{980368514637120000} - \tfrac{1262701803371\pi^2}{355704327287137332504000} \approx 1.26573 \cdot 10^{-9},\\
v_4^{(1)}(C_4) &= \tfrac{31874628962521753237}{105835701371904000000} - \tfrac{26003\pi^2}{1399680000} + \tfrac{610208\ln 2}{1913625} - \tfrac{536557\zeta(3)}{2592000} \approx 0.0021295294356,\\
v_4^{(3)}(C_4) &= \tfrac{19330626155629115959}{168272319220914585600000} - \tfrac{52276897\pi^2}{216801070940160000} + \tfrac{10004540239\ln 2}{77977156950000} - \tfrac{6155594561\zeta(3)}{73741860864000} \approx 7.5157 \cdot 10^{-8},\\
v_5^{(1)}(T_5) &= \tfrac{2207}{3265920} - \tfrac{244129\pi^2}{14522729760} + \tfrac{73522\pi^4}{541513323351} \approx 0.0005230827206879,\\
v_6^{(1)}(T_6) &= \tfrac{26609}{217818720} - \tfrac{3396146609\pi^2}{621871356506400} + \tfrac{1318349152898\pi^4}{12180206401298390455} \approx 0.00007880487647920397.
\end{aligned}$$

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