LEGENDRE POLYNOMIALS

& THEIR APPLICATION TO APPROXIMATIONS OF ELLIPSOIDAL FIGURES OF EQUILIBRIUM

[10 min] **A** ESSENTIALS

[50 min] **B** LEGENDRE POLYNOMIALS

[35 min] **C** SHAPE DESCRIPTION OF A NEAR-SPHERICAL BODY

D CONSTITUTIONAL AND EQUILIBRIUM BALANCE EQUATIONS

E WORKED EXAMPLES

A ESSENTIALS

- **Newton's approximation**: $(1+\varepsilon)^{\alpha} = 1 + \alpha\varepsilon + \frac{\alpha(\alpha-1)}{2}\varepsilon^2 + \dots$

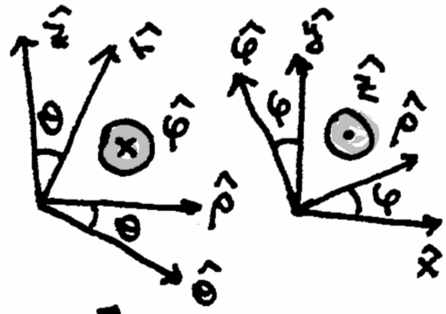
- **Vector identities**

$$\vec a \times \vec b = -\vec b \times \vec a \longrightarrow \vec a \times \vec a = 0$$

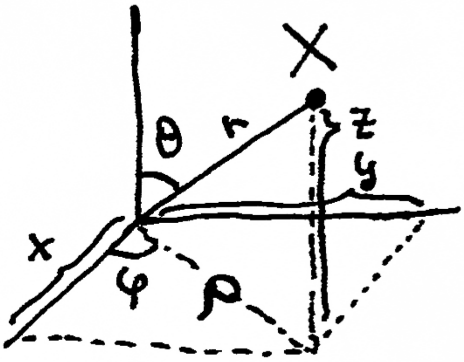
$$\vec a \cdot (\vec b \times \vec c) = \vec b \cdot (\vec c \times \vec a) = \vec c \cdot (\vec a \times \vec b)$$

$$\vec a \times (\vec b \times \vec c) = \vec b\,(\vec a \cdot \vec c) - \vec c\,(\vec a \cdot \vec b)$$

$$(T \times \vec b)\cdot \vec c = T \cdot (\vec b \times \vec c)$$



- **Coordinate systems**

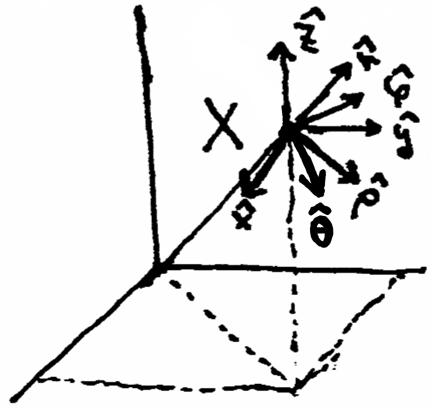


xyz

$\rho\varphi z$

$r\theta\varphi$

Scalar fields



$\hat x\hat y\hat z$

$\hat\rho\hat\varphi\hat z$

$\hat r\hat\theta\hat\varphi$

Unit vector fields

- **Diferential identities** (with coordinate fields)

$\nabla x = \hat x$	$\nabla \hat x = 0$	$\nabla\cdot\hat x = 0$
$\nabla y = \hat y$	$\nabla \hat y = 0$	$\nabla\cdot\hat y = 0$
$\nabla z = \hat z$	$\nabla \hat z = 0$	$\nabla\cdot\hat z = 0$
$\nabla \rho = \hat\rho$	$\nabla\hat\rho = \frac{1}{\rho}\hat\varphi\hat\varphi$	$\nabla\cdot\hat\rho = \frac{1}{\rho}$
$\nabla r = \hat r$	$\nabla\hat r = \frac{1}{r}(I - \hat r\hat r)$	$\nabla\cdot\hat r = \frac{2}{r}$
$\nabla\varphi = \frac{1}{\rho}\hat\varphi$	$\nabla\hat\varphi = \frac{-1}{\rho}\hat\varphi\hat\rho$	$\nabla\cdot\hat\varphi = 0$
$\nabla\theta = \frac{1}{r}\hat\theta$	$\nabla\hat\theta = \frac{\cot\theta}{r}\hat\varphi\hat\varphi - \frac{1}{r}\hat\theta\hat r$	$\nabla\cdot\hat\theta = \frac{\cot\theta}{r}$

$\nabla\times\hat x = \nabla\times\hat y = \nabla\hat z = 0$	$\Delta x = \Delta y = \Delta z = 0$	$\Delta\hat x = \Delta\hat y = \Delta\hat z = 0$
$\nabla\times\hat\rho = 0$	$\Delta\rho = \frac{1}{\rho}$	$\Delta\hat\rho = -\frac{1}{\rho^2}\hat\rho$
$\nabla\times\hat r = 0$	$\Delta r = \frac{2}{r}$	$\Delta\hat r = -\frac{2}{r^2}\hat r$
$\nabla\times\hat\varphi = \frac{\hat z}{\rho}$	$\Delta\varphi = 0$	$\Delta\hat\varphi = -\frac{1}{\rho^2}\hat\varphi$
$\nabla\times\hat\theta = \frac{\hat\varphi}{r}$	$\Delta\theta = \frac{1}{r^2}\cot\theta$	$\Delta\hat\theta = -\frac{1}{r^2}\left[2\cot\theta\,\hat r + \csc^2\theta\,\hat\theta\right]$

- <u>Differential identities</u> (Miscellaneous)

$$\nabla \times (\nabla \times \vec{u}) = \nabla \nabla \cdot \vec{u} - \Delta \vec{u} \qquad (\Delta := \nabla \cdot \nabla \text{ Laplacian})$$

$$\nabla \times \nabla T = 0 \qquad (T \text{ any tensor})$$

$$\nabla \times (\vec{u} \times \vec{v}) = \vec{u}\, \nabla \cdot \vec{v} - \vec{v}\, \nabla \cdot \vec{u} - \vec{u} \cdot \nabla \vec{v} + \vec{v} \cdot \nabla \vec{u}$$

$$\nabla (\phi T) = (\nabla \phi) T + \phi \nabla T \qquad (\phi \text{ a scalar field})$$

Tr

$$\Delta (\phi T) = (\Delta \phi) T + 2 \nabla \phi \cdot \nabla T + \phi \Delta T$$

$$\nabla \cdot (\vec{u} T) = (\nabla \cdot \vec{u}) T + \vec{u} \cdot \nabla T$$

$$\nabla \cdot (\phi \vec{u}) = (\nabla \phi) \cdot \vec{u} + \phi \nabla \cdot \vec{u}$$

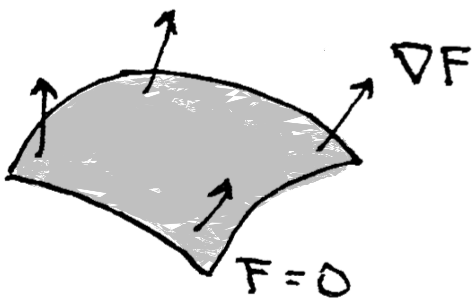
$$\Delta (\vec{r} \times \nabla \phi) = \vec{r} \times \nabla \Delta \phi$$

$$\nabla f(\phi) = f'(\phi) \nabla \phi \implies \nabla f(r) = f'(r) \hat{r}$$

$$\Delta f(r) = \frac{1}{r} (r f)'' = \frac{1}{r^2} (r^2 f')' \implies \Delta r^n = n(n+1) r^{n-2}$$

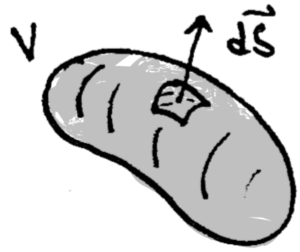
$$\hat{r} \cdot \nabla = \partial_r$$

- <u>Normal vector field</u> (to a given surface)

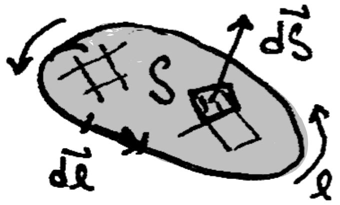


$$\boxed{\nabla F \perp F = 0}$$

- <u>Integral theorems</u>



$$\iiint_V dV\, \nabla T = \oiint_{\partial V} d\vec{S}\, T \qquad \text{Gauss'}$$



$$\iint_S d\vec{S} \times \nabla T = \oint_{\partial S} d\vec{l}\, T \qquad \text{Stokes'}$$

B LEGENDRE POLYNOMIALS

0 GENERAL ORTHOGONAL POLYNOMIALS

$$\underset{\text{leading coefficient}}{\phi_n(x) = k_n x^n} + \underset{\text{sub-leading coefficient}}{\tilde{k}_{n-1} x^{n-1}} + \dots ; \quad \overset{\text{scalar product}}{\langle \phi_n ; \phi_{n'} \rangle} := \int_M \phi_n(x)\, \bar{\phi}_{n'}(x)\, \overset{\text{weight}}{\rho(x)}\, dx = \|\phi_n\|^2 \delta_{nn'}$$

M interval (a,b) — ρ-norm — Kronecker δ

orthogonality relation

1 DEFINITIONS

- **Fundamental integral** : $F = \boxed{\int_{-1}^{1} f\left(x + \frac{t}{2}(1-x^2)\right) dx}$; $t \in [-1,1]$

  f analytic on $(-1,1)$

  I. Substitution $\quad y = x + \frac{t}{2}(1-x^2) \Rightarrow dx = \frac{dy}{\sqrt{1-2yt+t^2}}$

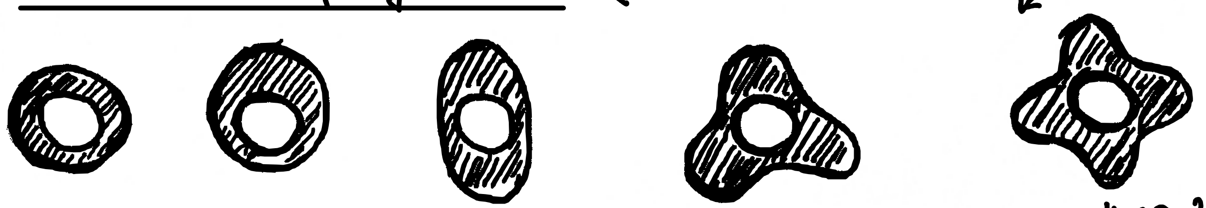
  $$\therefore \quad F = \int_{-1}^{1} \frac{f(y)}{\sqrt{1-2yt+t^2}}\, dy$$

  II Taylor expansion: $\quad f\left(x + \frac{t}{2}(1-x^2)\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \left[\frac{t}{2}(1-x^2)\right]^n$ $\quad \int$ by parts

  By parts integration: $\quad F = \int_{-1}^{1} f(x) \sum_{n=0}^{\infty} t^n \frac{(-1)^n}{2^n n!} \left[(1-x^2)^n\right]^{(n)}$

- **Rodrigues formula** : $\quad \boxed{P_n(x) := \frac{(-1)^n}{2^n n!} \left[(1-x^2)^n\right]^{(n)}}$

  * **Low order polynomials** $(x \equiv \hat{r} \cdot \hat{z} = \cos\theta)$



$P_0 = 1 \qquad P_1 = x \qquad P_2 = \frac{1}{2}(3x^2-1) \qquad P_3 = \frac{1}{2}(5x^3-3x) \qquad P_4 = \frac{1}{8}(35x^4-30x^2+3)$

  * **Leading term** : $P_n = \frac{(-1)^n}{2^n n!}\left[(-1)^n x^{2n} + \dots\right]^{(n)} = \underbrace{\frac{(2n)!}{2^n n!^2}}_{k_n} x^n + \dots$

- **Generating function** (GF):

  By comparing I & II in F : $\quad \boxed{G(x,t) := \sum_{n=0}^{\infty} P_n(x)\, t^n = \frac{1}{\sqrt{1-2xt+t^2}}}$; $t \in [-1;1]$

2 BASIC PROPERTIES

$$G(x,t)=\sum_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}$$

- <u>Special values</u> ($\hookrightarrow$ GF)

a) $x=1$: $\frac{1}{1-t} = \sum P_n(1)\, t^n \quad \therefore P_n(1)=1$

b) $x=-1$: $\frac{1}{1+t} = \sum P_n(-1)\, t^n \quad \therefore P_n(-1) = (-1)^n$

in general $G(-x,-t) = G(x,t) \quad \therefore P_{2n}(-x) = P_{2n}(x) \ \wedge\ P_{2n+1}(-x) = -P_{2n+1}(x)$

c) $x=0$: $\frac{1}{\sqrt{1+t^2}} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{(2\ell)!!}(-t^2)^\ell \equiv \sum_{n=0}^{\infty} P_n(0)\, t^n$

$$\therefore P_{2\ell+1}(0)=0 \ \wedge\ P_{2\ell}(0) = (-1)^\ell \frac{(2\ell-1)!!}{(2\ell)!!}$$

Alternatively from Rodrigues: $n = 2\ell$, so $\qquad P_{2\ell+1}(0)=0$ (when n is odd we have no x^n term)

$$(1-x^2)^n = \dots + (-1)^\ell \binom{2\ell}{\ell} x^{2\ell} + \dots$$

$$\left[(1-x^2)^n\right]^{(n)}\Big|_{x=0} = (-1)^\ell \binom{2\ell}{\ell}(2\ell)! \quad \therefore P_{2\ell}(0) = (-1)^\ell \frac{(2\ell)!}{4^\ell \ell!^2} = (-1)^\ell \frac{(2\ell-1)!!}{(2\ell)!!}$$

- <u>Polynomial orthogonality relation</u> : WLOG $n \geq n'$:

$*\ \langle P_n, P_{n'}\rangle = \int_{-1}^{1} P_n P_{n'}\, dx \overset{RG}{=} \frac{(-1)^n}{2^n n!}\int_{-1}^{1} \left[(1-x^2)^n\right]^{(n)} \underbrace{P_{n'}}_{k_{n'}x^{n'}+\dots}\, dx =$ $\int$ by parts

$$\frac{k_n \cancel{n!}\,\delta_{nn'}}{2^n \cancel{n!}} \int_{-1}^{1} (1-x^2)^n\, dx \overset{\dots}{=} \frac{2}{2n+1}\delta_{nn'}$$

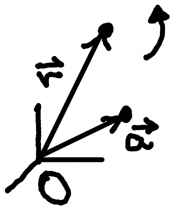
$$\boxed{\int_{-1}^{1} P_n P_{n'}\, dx = \frac{2}{2n+1}\delta_{nn'}} \quad \text{(POG)}$$

$*\ \sum_{n=0}^{\infty} \|P_n\|^2 t^{2n} = \int_{-1}^{1} \sum_{n,n'} \langle P_n, P_{n'}\rangle\, t^n t^{n'}\, dx$

$$\overset{GF}{=} \int_{-1}^{1} \left(\frac{1}{\sqrt{1-2xt+t^2}}\right)^2 dx = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n} \quad \therefore \|P_n\|^2 = \frac{2}{2n+1}$$

- <u>Coulomb kernel expansion</u>

$$\frac{1}{|\vec r - \vec a|} = \frac{1}{\sqrt{r^2 - 2\vec r\cdot\vec a + a^2}} = \frac{1}{r}\frac{1}{\sqrt{1-2\frac{a}{r}\hat r\cdot\hat a + \left(\frac{a}{r}\right)^2}} \overset{\text{by GF}}{=} \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n P_n(\hat r\cdot\hat a)\ ;\ r \geq a$$



for $r \leq a$, we replace $r \leftrightarrow a$: $\frac{1}{|\vec r - \vec a|} = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\hat r\cdot\hat a)$

$$\therefore \frac{1}{|\vec r - \vec a|} = \sum_{n=0}^{\infty} g_n(r,a)\, P_n(\hat r\cdot\hat a) \quad \text{where } g_n(r,a) = \begin{cases} \frac{1}{r}\left(\frac{a}{r}\right)^n ; & r \geq a \\ \frac{1}{a}\left(\frac{r}{a}\right)^n ; & r \leq a \end{cases}$$

- Laplacian

* $\Delta r^n = n(n+1)r^{n-2} \Rightarrow \Delta\frac{1}{r} \overset{r\neq 0}{=} 0 \overset{\text{transl.}}{\Rightarrow} \Delta\frac{1}{|\vec{r}-\vec{a}|} = 0;\ \vec{r}\neq\vec{a}$

* In Coulomb's expansion, taking Δ and comparing the a^n and a^{-n-1} terms, respectively:

$$\Delta(r^n P_n) = 0 \quad \wedge \quad \Delta(r^{-n-1}P_n) \overset{r\neq 0}{=} 0$$

* On the other hand, by $\Delta(fg) = (\Delta f)g + 2\nabla f\cdot\nabla g + f\Delta g$:

$$\Delta(r^n P_n) = (\Delta r^n)P_n + 2\nabla r^n\cdot\nabla P_n + r^n\Delta P_n =$$
$$= n(n+1)r^{n-2}P_n + \cancel{2nr^{n-1}\underbrace{\hat{r}\cdot\nabla P_n}_{\partial_r}}^{0} + r^n\Delta P_n = 0$$

$\therefore$ $\boxed{\Delta P_n = -\frac{1}{r^2}n(n+1)P_n}$ for any $\hat{a}$

|| Exercise: Show that $\Delta(r^{n+2}P_n) = 2(2n+3)r^n P_n$

3 ORTHOGONAL FORMULAE

- Two legendres orthogonality

* Recall $\oint_{\partial V} d\vec{S}\cdot T = \int_V \nabla\cdot T\, dV$ Gauss' theorem, put

$T = f\nabla g - g\nabla f \Rightarrow \nabla\cdot T = \cancel{\nabla f\cdot\nabla g} + f\Delta g - \cancel{\nabla g\cdot\nabla f} - g\Delta f$

$\therefore$ We get Green's identity $\oint_{\partial V}(f\nabla g - g\nabla f)\cdot d\vec{S} = \int_V f\Delta g - g\Delta f\, dV$

* Let $V = B_1(0)$ (unit ball in $\mathbb{E}^3$), $f = r^n P_n(\hat{r}\cdot\hat{a})$; $g = r^m P_m(\hat{r}\cdot\hat{b})$

$\therefore$ RHS OF GI: $\int_V f\Delta g - g\Delta f\, dV = 0$

* Since $d\vec{S} = \hat{r}r^2 d\Omega = \hat{r}\,d\Omega$ and $\nabla P_n(\hat{r}\cdot\hat{a}) \perp \hat{r}$,

we get $(f\nabla g - g\nabla f)\cdot d\vec{S} = (m-n)P_n(\hat{r}\cdot\hat{a})P_m(\hat{r}\cdot\hat{b})$

$\text{LHS} \overset{GI}{=} \text{RHS} \Rightarrow \oint P_n(\hat{r}\cdot\hat{a})P_m(\hat{r}\cdot\hat{b})\,d\Omega = 0$ for $n\neq m$

• <u>Dirac kernel expansion</u> $(\delta(\vec r - \vec a))$

$*$ Recall $\Delta(\frac{1}{r}) = 0$ at $r \neq 0 \Rightarrow$ Ansatz $\Delta \frac{1}{r} = A\,\delta(\vec r)$

Integrating over $B_1(0)$: $\int_V A\,\delta(\vec r)\,dV = A$, on the o.h.

$$\text{LHS} = \int_V \Delta \frac{1}{r}\,dV = \int_V \nabla\cdot\nabla \frac{1}{r}\,dV \overset{GT}{=} \oint_S \left(\nabla \frac{1}{r}\right)\cdot d\vec S =$$

$$= -\oint \frac{1}{r^2}\,\hat r \cdot r^2 \hat r\,d\Omega = -\oint d\Omega = -4\pi \quad \therefore\ A = -4\pi$$

and $\Delta \frac{1}{r} = -4\pi\,\delta(\vec r) \overset{\text{transl.}}{\Longrightarrow} \boxed{\Delta \frac{1}{|\vec r - \vec a|} = -4\pi\,\delta(\vec r - \vec a)}$

$*$ Taking Δ of the Coulomb's expansion:

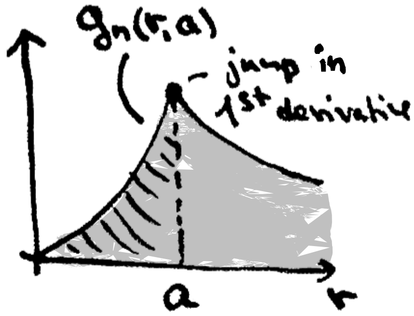
$$-4\pi\,\delta(\vec r - \vec a) = \sum_{n=0}^{\infty} \Delta\left(g_n(r,a)\,P_n(\hat r\cdot\hat a)\right) =$$

$$= \sum_{n=0}^{\infty} (\Delta g_n)P_n + \cancelto{0}{2\nabla g_n \cdot \nabla P_n} + g_n \Delta P_n =$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{r^2}(r^2 g_n')' - \frac{1}{r^2} g_n\, n(n+1)\right) P_n$$

$\Leftrightarrow$ only possible if $(r^2 g_n')' - g_n n(n+1) = B\,\delta(r-a)$

$*$ Integrating over $\int_{a-\varepsilon}^{a+\varepsilon} dr$: $a^2\left(g_n'(a^+) - g_n'(a^-)\right) = B$



But $g_n'(a^-) = \left[\frac{1}{a}\left(\frac{r}{a}\right)^n\right]'\Big|_a = \frac{n}{a^2}$

$g_n'(a^+) = \left[\frac{1}{r}\left(\frac{a}{r}\right)^n\right]'\Big|_a = -\frac{n+1}{a^2}$

$$\therefore \quad \boxed{\delta(\vec r - \vec a) = \frac{1}{a^2}\,\delta(r-a)\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} P_n(\hat r\cdot\hat a)} \quad \text{(DKE)}$$

- <u>Master orthogonal formula</u>

* Two ways how to compute $I = \int_{\mathbb{R}^3} P_m(\hat{r}\cdot\hat{b})\,\delta(\vec{r}-\vec{a})\,dV$

* Either by def of δ: $I \overset{\vec{r}=\vec{a}}{=} P_m(\hat{a}\cdot\hat{b})$

* Or by DKE: $I = \oint_\Omega d\Omega \int_0^\infty dr\, r^2\, P_m(\hat{r}\cdot\hat{b})\,\delta(\vec{r}-\vec{a}) =$

$$= \frac{1}{a^2}\underbrace{\int_0^\infty r^2\delta(r-a)\,dr}_{a^2}\;\sum_{n=0}^{\infty}\frac{2n+1}{4\pi}\underbrace{\oint d\Omega\, P_n(\hat{r}\cdot\hat{a})\,P_m(\hat{r}\cdot\hat{b})}_{=0 \text{ if } n\neq m}$$

$$= \frac{2m+1}{4\pi}\oint P_m(\hat{r}\cdot\hat{a})\,P_m(\hat{r}\cdot\hat{b})\,d\Omega$$

* $\therefore$ $$\boxed{\oint_\Omega P_n(\hat{r}\cdot\hat{a})\,P_m(\hat{r}\cdot\hat{b})\,d\Omega = \frac{4\pi}{2n+1}\,\delta_{nm}\,P_n(\hat{a}\cdot\hat{b})}\quad (\mathrm{OG})$$

- <u>Polynomial orthogonal formula</u>

* Spec case $\hat{a}=\hat{b}=\hat{z} \longrightarrow \hat{r}\cdot\hat{a} = \hat{r}\cdot\hat{b} = \cos\theta \equiv x,$

* $\mathrm{LHS} = \int_0^{2\pi}\int_0^{\pi} P_n(\cos\theta)\,P_m(\cos\theta)\sin\theta\,d\theta\,d\varphi = 2\pi\int_{-1}^{1} P_n(x)\,P_m(x)\,dx$

* $\mathrm{RHS} = \frac{4\pi}{2n+1}\,\delta_{nm}\,\underbrace{P_n(1)}_{=1}$ $\quad\therefore\quad \boxed{\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{2}{2n+1}\,\delta_{nm}}$

- <u>φ – orthogonal formula</u>

* let $\delta(x-y) = \sum_{n=0}^{\infty} A_n P_n(x) \;\Big/ \int_{-1}^{1} dx\, P_m(x)$: $P_n(y) = \frac{2A_n}{2n+1}$

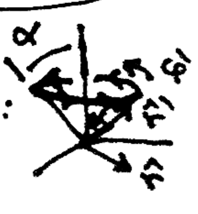
$\therefore\; \delta(x-y) = \sum_{n=0}^{\infty}\frac{2n+1}{2}\,P_n(x)\,P_n(y)$

* Put $x = \hat{r}'\cdot\hat{z} = \cos\theta'$, $y = \cos\alpha$ (const.)

$$\delta(\cos\theta' - \cos\alpha) = \sum \frac{2n+1}{2}\,P_n(\hat{r}'\cdot\hat{z})\,P_n(\cos\alpha) \;\Big/ \oint d\Omega'\, P_m(\hat{r}'\cdot\hat{r})$$

$$\int_0^{\pi}\int_0^{2\pi}\delta(c\theta' - c\alpha)\,s\theta'\,P_n(\hat{r}'\cdot\hat{r})\,d\varphi'\,d\theta' \overset{\mathrm{OG}}{=} \frac{2n+1}{2}\,\frac{4\pi}{2n+1}\,P_n(\hat{r}\cdot\hat{z})\,P_n(c\alpha)$$

$$\therefore\; \boxed{\int_0^{2\pi} P_n(\hat{r}'\cdot\hat{r})\,d\varphi' = 2\pi\,P_n(\hat{r}\cdot\hat{z})\,P_n(\hat{z}\cdot\hat{r}')}\quad (\varphi\mathrm{OG}):$$



4 VECTOR LEGEDRES

- <u>Axial expansion</u> :

  * Scalar fields $\phi(r,\theta) = \sum \phi_n(r) P_n$ — argument $\hat{r}\cdot\hat{z} = c\theta$

  * Vector fields $\vec{\psi}(r,\theta) = \sum U_n(r)\underbrace{\hat{r}P_n}_{\vec{P}_n} + V_n(r)\underbrace{r\nabla P_n}_{\vec{Q}_n} + W_n(r)\underbrace{\vec{r}\times\nabla P_n}_{\vec{R}_n}$

  vector legendres $\longrightarrow$ $\vec{P}_n$, $\vec{Q}_n$, $\vec{R}_n$

- <u>Low order values</u> :

  * $P_0 = 1 \Rightarrow \vec{Q}_0 = \vec{R}_0 = 0$

  * $\vec{Q}_1 = r\nabla P_1 = r\nabla(\hat{r}\cdot\hat{z}) = r\nabla\hat{r}\cdot\hat{z} = r\,\frac{I-\hat{r}\hat{r}}{r}\cdot\hat{z} =$

  $= \hat{z} - \hat{r}(\hat{r}\cdot\hat{z}) = (\hat{r}\hat{r} + \underbrace{\hat{\theta}\hat{\theta}}_{+\hat{\phi}\hat{\phi}})\cdot(\hat{z} - \hat{r}(\hat{r}\cdot\hat{z})) =$

  $= \hat{\theta}(\hat{\theta}\cdot\hat{z}) = -s\theta\,\hat{\theta}$

  * $\vec{R}_1 = \hat{r}\times\vec{Q}_1 = -s\theta\,\hat{r}\times\hat{\theta} = -s\theta\,\hat{\phi}$

$\vec{Q}_1$

$\vec{R}_1$

- <u>Basic properties</u>

  a) $(\vec{P}_n, \vec{Q}_n, \vec{R}_n)$ forms an orthogonal basis $\Rightarrow \hat{r}\cdot\vec{Q}_n = \hat{r}\cdot\vec{R}_n = 0$

  b) Since $\vec{P}_n, \vec{Q}_n, \vec{R}_n$ indep. of $r \Rightarrow \partial_r = \hat{r}\cdot\nabla = 0$

  c) Divergence :

  * $\nabla\cdot\vec{P}_n = \nabla\cdot(\hat{r}P_n) = \cancel{\hat{r}\cdot\nabla P_n}^{0} + \frac{2}{r}P_n = \frac{2}{r}P_n$

  * $\nabla\cdot\vec{Q}_n = \nabla\cdot(r\nabla P_n) = \cancel{\nabla r\cdot\nabla P_n}^{0} + r\Delta P_n = -\frac{1}{r}n(n+1)P_n$

  * $\nabla\cdot\vec{R}_n = \nabla\cdot(\vec{r}\times\nabla P_n) = \nabla P_n\cdot\cancel{(\nabla\times\vec{r})}^{0} - \vec{r}\cdot\cancel{(\nabla\times\nabla P_n)}^{0} = 0$

  $[\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{c}\times\vec{a})]$

  d) Transposed $\hat{r}\cdot\nabla$:

  * $\hat{r}\cdot(\nabla\vec{P}_n)^T = (\nabla\vec{P}_n)\cdot\hat{r} = \nabla(\hat{r}P_n)\cdot\hat{r} = \nabla P_n\,\hat{r}\cdot\hat{r} + P_n\cancel{\nabla\hat{r}\cdot\hat{r}}^{0} = \frac{\vec{Q}_n}{r}$

  * $\nabla\vec{Q}_n\cdot\hat{r} = \nabla(r\nabla P_n)\cdot\hat{r} = \hat{r}\,\cancel{\nabla P_n\cdot\hat{r}}^{0} + r\nabla^2 P_n\cdot\hat{r} \overset{\text{sym}}{=}$

  $= r\hat{r}\cdot\nabla^2 P_n = r\,\partial_r\nabla P_n = r\,\partial_r\frac{\vec{Q}_n}{r} = -\frac{1}{r}\vec{Q}_n$

$$* \; \nabla \vec{R}_n \cdot \hat{r} = \nabla(\vec{r} \times \nabla P_n) \cdot \hat{r} = (I \times \nabla P_n) \cdot \hat{r} - (\nabla^2 P_n \times \vec{r}) \cdot \hat{r} =$$

$$= I \cdot (\nabla P_n \times \hat{r}) - \nabla^2 P_n \cdot \cancel{(\vec{r} \times \hat{r})}^{0} = \nabla P_n \times \hat{r} = -\frac{1}{r} \vec{R}_n$$

e) „x" with $\hat{r}$:

$$* \; \hat{r} \times \vec{P}_n = \hat{r} \times \hat{r} P_n = \vec{O}$$

$$* \; \hat{r} \times \vec{Q}_n = \hat{r} \times r \nabla P_n = \vec{r} \times \nabla P_n = \vec{R}_n$$

$$* \; \hat{r} \times \vec{R}_n = \hat{r} \times (\vec{r} \times \nabla P_n) = \hat{r}\,\cancel{(\hat{r} \cdot \nabla P_n)}^{0} - \nabla P_n (\hat{r} \cdot \vec{r}) = -\vec{Q}_n$$

f) Curl:

$$* \; \nabla \times \vec{P}_n = \nabla \times (\hat{r} P_n) = \nabla P_n \times \hat{r} + \cancel{(\nabla \times \hat{r})}^{0} P_n = -\frac{1}{r} \vec{R}_n$$

$$* \; \nabla \times \vec{Q}_n = \nabla \times (r \nabla P_n) = \hat{r} \times \nabla P_n + r \nabla \times \nabla P_n = \frac{1}{r} \vec{R}_n$$

$$* \; \nabla \times \vec{R}_n = \nabla \times (\vec{r} \times \nabla P_n) = \vec{r} \Delta P_n + \nabla P_n \cdot \nabla \vec{r} - \nabla P_n \nabla \cdot \vec{r} - \vec{r} \cdot \nabla^2 P_n$$

$$= -\frac{n(n+1)}{r} \vec{P}_n + \nabla P_n - 3 \nabla P_n + \frac{1}{r} \vec{Q}_n = -\frac{1}{r} (\vec{Q}_n + n(n+1) \vec{P}_n)$$

g) Laplacian

$$* \; \Delta \vec{P}_n = \Delta(\hat{r} P_n) = \Delta \hat{r}\, P_n + 2 \nabla P_n \cdot \nabla \hat{r} + \hat{r} \Delta P_n =$$

$$= -\frac{2}{r^2} \hat{r} P_n + \frac{2}{r} \nabla P_n \cdot (I - \hat{r}\hat{r}) - \frac{\hat{r}}{r^2} n(n+1) P_n$$

$$= -\frac{2}{r^2} \vec{P}_n + \frac{2}{r^2} (\vec{Q}_n - \vec{O}) - \frac{n(n+1)}{r^2} \vec{P}_n = -\frac{2 + n(n+1)}{r^2} \vec{P}_n + \frac{2 \vec{Q}_n}{r^2}$$

$$* \; \Delta \vec{Q}_n = \Delta(r \nabla P_n) = \Delta r \, \nabla P_n + 2 \nabla r \cdot \nabla^2 P_n + r \nabla \Delta P_n =$$

$$= \frac{2}{r} \nabla P_n + 2 \hat{r} \cdot \nabla^2 P_n - n(n+1) r \nabla \left(\frac{1}{r^2} P_n \right) =$$

$$\overset{\dots}{=} \frac{n(n+1)}{r^2} [2 \vec{P}_n + \vec{Q}_n]$$

$$* \; \Delta \vec{R}_n = \Delta(\vec{r} \times \nabla P_n) = \vec{r} \times \nabla \Delta P_n = -n(n+1) \vec{r} \times \nabla \left(\frac{1}{r^2} P_n \right) =$$

$$= -\frac{n(n+1)}{r^2} \vec{r} \times \nabla P_n = -\frac{n(n+1)}{r^2} \vec{R}_n$$

‖ Exercise: Show $(n+1) \nabla (r^n P_n) + \nabla \times (r^n \vec{R}_n) = 0$.

5 Miscellaneous

a) $P_1^2 \overset{\text{even}}{=} AP_0 + BP_2$ $\qquad (P_n = P_n(\hat{r}\cdot\hat{z}) = P_n(\cos\theta))$

$* \ \frac{1}{4\pi}\oint : \ \frac{1}{3} \equiv A$

$* \ c\theta = 1 : \ 1 \equiv A + B$

$$\boxed{P_1^2 = \frac{1}{3}P_0 + \frac{2}{3}P_2}$$

b) $s^2\theta = 1 - c^2\theta = 1 - P_1^2 \quad \therefore \quad \boxed{s^2\theta = \frac{2}{3}(P_0 - P_2)}$

c) $\vec{Q}_1 = -s\theta\,\hat{\theta} \Rightarrow \boxed{\underbrace{\vec{Q}_1\cdot\vec{Q}_1}_{s^2\theta} = \frac{2}{3}(P_0 - P_2)}$

d) $\underbrace{}_{\text{Since}} P_1^2 = \frac{1}{3}P_0 + \frac{2}{3}P_2 \ /\nabla : \ 2P_1\vec{Q}_1 = \frac{2}{3}\vec{Q}_2 \ \therefore \ \boxed{P_1\vec{Q}_1 = \frac{1}{3}\vec{Q}_2}$

e) $P_1P_2 \overset{\text{odd}}{=} AP_1 + BP_3 \qquad /\cdot P_1$

$* \ \frac{1}{4\pi}\oint \underbrace{P_1^2}_{\frac{1}{3}P_0+\frac{2}{3}P_2} P_2\, d\Omega = \frac{2}{3}\cdot\frac{1}{5} \overset{\text{RHS}}{\equiv} \frac{1}{4\pi}\oint P_1(AP_1 + BP_3)\,d\Omega = \frac{1}{3}A$

$A = \frac{2}{5}$

$* \ c\theta = 1 \ : \ 1 = A + B \quad \therefore \quad \boxed{P_1P_2 = \frac{2}{5}P_1 + \frac{3}{5}P_3}$

f) $\vec{Q}_1\cdot\vec{Q}_2 = 3\vec{Q}_1\cdot(P_1\vec{Q}_1) = 3P_1\,\frac{2}{3}(P_0 - P_2) = 2(P_1 - \frac{2}{5}P_1 - \frac{3}{5}P_3)$

$\therefore \ \boxed{\vec{Q}_1\cdot\vec{Q}_2 = \frac{6}{5}(P_1 - P_3)}$

g) $P_1^3 \overset{\text{odd}}{=} AP_1 + BP_3 \quad /\cdot P_1$

$* \ \frac{1}{4\pi}\oint P_1^4\,d\Omega = \frac{1}{4\pi}\oint (\underbrace{P_1^2}_{\frac{1}{3}P_0+\frac{2}{3}P_2})^2\,d\Omega = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\frac{1}{5} \equiv \underbrace{\frac{1}{4\pi}\oint P_1(AP_1 + BP_3)\,d\Omega}_{A/3}$

$* \ c\theta = 1 \ : \ 1 \equiv A + B \quad \therefore \quad \boxed{P_1^3 = \frac{3}{5}P_1 + \frac{2}{5}P_3}$

h) Taking ∇ of g): $3\underbrace{P_1^2}_{\frac{1}{3}P_0+\frac{2}{3}P_2}\vec{Q}_1 = \frac{3}{5}\vec{Q}_1 + \frac{2}{5}\vec{Q}_2$

$\therefore \quad \boxed{P_2\vec{Q}_1 = \frac{1}{5}(\vec{Q}_3 - \vec{Q}_1)}$

i) Since $P_1P_2 = \frac{2}{5}P_1 + \frac{3}{5}P_3 \xrightarrow{\nabla} P_2\vec{Q}_1 + P_1\vec{Q}_2 = \frac{2}{5}\vec{Q}_1 + \frac{3}{5}\vec{Q}_3$

& since $P_2\vec{Q}_1$ in h) : $\boxed{P_1\vec{Q}_2 = \frac{3}{5}\vec{Q}_1 + \frac{2}{5}\vec{Q}_3}$

j) $P_1^2 P_2 \overset{\text{even}}{=} AP_0 + BP_2 + CP_4$

$$* \ \frac{1}{4\pi}\oint \underbrace{P_1^2}_{\frac{1}{3}P_0+\frac{2}{3}P_2} P_2 \, d\Omega = \frac{2}{3}\frac{1}{5} \equiv A \quad \therefore A = \frac{2}{15} \qquad \cdot P_2$$

$$* \ \frac{1}{4\pi}\oint P_1^2 P_2^2 \, d\Omega = \frac{1}{4\pi}\oint (\underbrace{P_1 P_2}_{\frac{2}{5}P_1+\frac{3}{5}P_3})^2 d\Omega = \left(\frac{2}{5}\right)^2\frac{1}{3} + \left(\frac{3}{5}\right)^2\frac{1}{7} \equiv \frac{B}{5}$$

$$* \ c\theta = 1: \ 1 \equiv A + B + C \qquad \therefore \boxed{P_1^2 P_2 = \frac{2}{15}P_0 + \frac{11}{21}P_2 + \frac{12}{35}P_4}$$

k) $\vec{Q}_2 \cdot \vec{Q}_2 = 3P_1\vec{Q}_1 \cdot 3P_1\vec{Q}_1 = 6P_1^2(P_0 - P_2) = 6(A^2 - P_1^2 P_2) =$

$$= 6\left(\frac{1}{3}P_0 + \frac{2}{3}P_2 - \frac{2}{15}P_0 - \frac{11}{21}P_2 - \frac{12}{35}P_4\right) \therefore \boxed{\vec{Q}_2\cdot\vec{Q}_2 = 6\left(\frac{1}{5}P_0 + \frac{1}{7}P_2 - \frac{12}{35}P_4\right)}$$

l) $c\theta^4 = P_1^4 = P_1^2 P_1^2 = P_1^2\left(\frac{1}{3}P_0 + \frac{2}{3}P_2\right) = \frac{1}{3}\left(P_1^2 + 2P_1^2 P_2\right)$

$$= \frac{1}{3}\left(\frac{1}{3}P_0 + \frac{2}{3}P_2 + 2\left(\frac{2}{15}P_0 + \frac{11}{21}P_2 + \frac{12}{35}P_4\right)\right) \therefore \boxed{P_1^4 = \frac{1}{5}P_0 + \frac{4}{7}P_2 + \frac{8}{35}P_4}$$

m) $s^4\theta = (1 - c^2\theta)^2 = (1 - P_1^2)^2 = 1 - 2P_1^2 + P_1^4 = 1 - 2\left(\frac{1}{3}P_0 + \frac{2}{3}P_2\right)$

$$+ \frac{1}{5}P_0 + \frac{4}{7}P_2 + \frac{8}{35}P_4 \quad \therefore \boxed{s^4\theta = \frac{8}{15}P_0 - \frac{16}{21}P_2 + \frac{8}{35}P_4}$$

n) Since $P_1^2 = \frac{1}{3}P_0 + \frac{2}{3}P_2 \Rightarrow P_2 = \frac{1}{2}(3P_1^2 - 1)$ (familiar!)

$$* \ P_2^2 = \frac{1}{2}P_2(3P_1^2 - 1) = \frac{1}{2}(3P_1^2 P_2 - P_2) =$$

$$= \frac{1}{2}\left[3\left(\frac{2}{15}P_0 + \frac{11}{21}P_2 + \frac{12}{35}P_4\right) - P_2\right] \therefore \boxed{P_2^2 = \frac{1}{5}P_0 + \frac{2}{7}P_2 + \frac{18}{35}P_4}$$

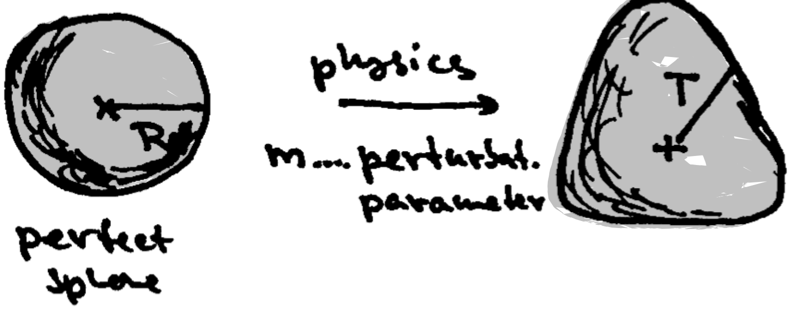
o) Taking ∇: $2P_2\vec{Q}_2 = \frac{1}{5}\cancel{\vec{Q}_0}^{\,0} + \frac{2}{7}\vec{Q}_2 + \frac{18}{35}\vec{Q}_4$

$$\therefore \boxed{P_2\vec{Q}_2 = \frac{1}{7}\vec{Q}_2 + \frac{9}{35}\vec{Q}_4}$$

C SHAPE DESCRIPTION OF A NEAR-SPHERICAL BODY

1 TOPOGRAPHY & ITS TRANSFORMATIONS

- Description



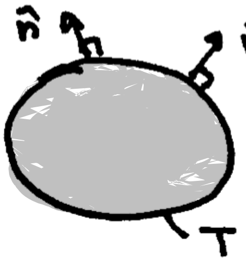
Topography ($O(m^2)$)

$$T = R(1 + mt + m^2 u)$$

$$t = \sum_{n=0}^{\infty} t_n P_n$$

$$u = \sum_{n=0}^{\infty} u_n P_n$$

- Normal vector field



$\hat{n} \perp r - T = 0 \perp \vec{n} := \nabla(r - T) = \hat{r} - Rm\nabla t - Rm^2 \nabla u$

so $|\vec{n}| = \sqrt{\vec{n}\cdot\vec{n}} = \sqrt{1 + m^2R^2 \ (\nabla t \cdot \nabla t)} =$

$$= 1 + \tfrac{1}{2} m^2 R^2 |\nabla t|^2 + O(m^3)$$

$$\therefore \ \hat{n} = \frac{\vec{n}}{|\vec{n}|} = (\hat{r} - mR\nabla t - m^2 R\nabla u)(1 - \tfrac{1}{2}R^2 m^2 |\nabla t|^2) =$$

$$= \hat{r} - mR\nabla t - \tfrac{1}{2} m^2 R^2 \hat{r} |\nabla t|^2 - m^2 R \nabla u + O(m^3)$$

- Fixed-volume topography constraints

$$V = \iiint dV = \oint d\Omega \int_0^T dr\, r^2 = \frac{R^3}{3} \oint (1 + mt + m^2 u)^3 d\Omega =$$

$$= \frac{R^3}{3} \oint (1 + 3mt + 3m^2 u + 3m^2 t^2)\, d\Omega =$$

$$= \underbrace{\frac{4\pi R^3}{3}}_{V_0} \Big(1 + \underbrace{3m t_0}_{\delta} + 3m^2 u_0 + 3m^2 \sum_{n=0}^{\infty} \frac{t_n^2}{2n+1} \Big)$$

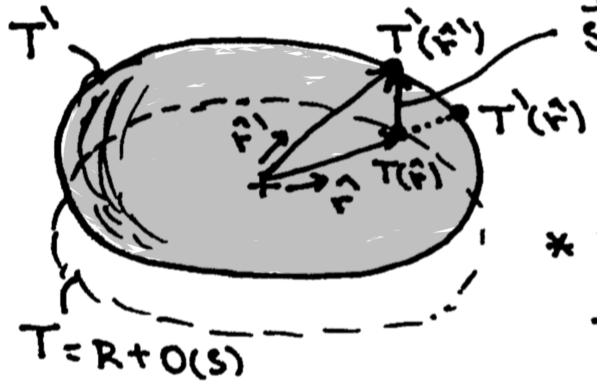
δ stretch factor $\left[\delta = \frac{V}{V_0} - 1 + O(m^2)\right]$

* Constraints on $V = V_0$

a) $t_0 = 0$

b) $u_0 = -\sum_{n=0}^{\infty} \frac{t_n^2}{2n+1} = -\frac{t_1^2}{3} - \frac{t_2^2}{5} - \frac{t_3^2}{7} - \dots$

- **Shifted topography**



$\vec s = s\hat z$ translation ($s=(m\alpha+m^2\beta)R$) of old topo $T=T(\hat r)=R(1+mt+m^2u)$ to new topo $T'=T'(\hat r)=R(1+mt'+m^2u')$

* Note $T'(\hat r')\hat r' = T(\hat r)\hat r + \vec s \quad /|\cdot|$

$$T'(\hat r') = \sqrt{T^2+2Ts\underbrace{\hat r\cdot\hat z}_{P_1}+s^2} =$$

$$= T\sqrt{1+2\frac{s}{T}P_1+\frac{s^2}{T}} \approx T\left(1+\frac{s}{T}P_1+\frac{s^2}{2T^2}-\frac{s^2}{2T^2}P_1^2\right) =$$

$$= T + sP_1 + \frac{s^2}{3R}(1-P_2) + O(s^3) =$$

$$= R\left[1+m(t+\alpha P_1)+m^2\left(u+\beta P_1+\frac{\alpha^2}{3}(1-P_2)\right)\right] + O(m^3)$$

* Transformation of the $\hat r$ vector $(O(m))$

$$\hat r' = \frac{T\hat r+\vec s}{|T\hat r+\vec s|} = \frac{T\hat r+\vec s}{T'(\hat r')} = (T\hat r+\vec s)(T+sP_1)^{-1} =$$

$$= \left(\hat r+\frac{\vec s}{T}\right)\left(1+\frac{s}{T}P_1\right)^{-1} = \left(\hat r+\frac{\vec s}{R}\right)\left(1-\frac{s}{R}P_1\right) =$$

$$= \hat r + \frac{s}{R}(\hat z-\hat r P_1) = \hat r + \frac{s}{R}\vec\Theta_1 = \hat r + m\alpha\vec\Theta_1 + O(m^2)$$

* Transformation $T'(\hat r') \longrightarrow T'(\hat r) \equiv T'$:

$$R(1+mt'+m^2u') \equiv T' = T'(\hat r) = T'(\hat r' - \alpha m\vec\Theta_1 + O(m^2)) =$$

$$= T'(\hat r') - (\alpha m\vec\Theta_1 + O(m^2))\cdot r\nabla T'(\hat r')$$

But $(\alpha m\vec\Theta_1 + O(m^2))\cdot r\nabla T'(\hat r') = (\alpha m\vec\Theta_1 + O(m^2))\cdot Rm(r\nabla t + \alpha\vec\Theta_1)$

$$= R\alpha m^2\vec\Theta_1\cdot r\nabla t + R\alpha^2 m^2\underbrace{\vec\Theta_1\cdot\vec\Theta_1}_{\frac{2}{3}(1-P_2)}$$

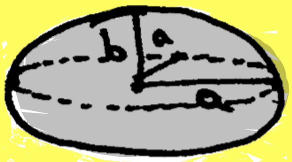
$$\therefore\ R[1+mt'+m^2u'] \equiv R\left[1+m(t+\alpha P_1)+m^2\left(u+\beta P_1-\frac{\alpha^2}{3}(1-P_2)-\alpha\vec\Theta_1\cdot r\nabla t\right)\right]$$

$$\therefore\ \boxed{a)\ t' = t+\alpha P_1} \quad \boxed{b)\ u' = u+\beta P_1 - \frac{\alpha^2}{3}(1-P_2) - \alpha\vec\Theta_1\cdot r\nabla t}$$

* Shift degeneracy (in problems with $\hat z$-transl. symmetry)

$\Rightarrow$ we can choose $t_1 = 0,\ u_1 = 0$ $(\because \alpha, \beta)$

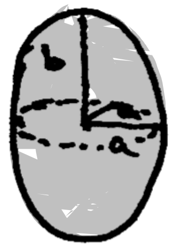
② BIAXIAL ELLIPSOIDS (SPHEROIDS)



Oblate spheroid

$a > b$

vs.



prolate spheroid

$a < b$

$$\left.\right\} \quad \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \text{or} \quad \frac{\rho^2}{a^2} + \frac{z^2}{b^2} = 1$$

- <u>Topographic expansion</u> (oblate sph. only) ($O(m^2)$)

* Excentricity $e^2 := 1 - \frac{b^2}{a^2}$

* Volume & axes

$$V = \frac{4}{3}\pi a^2 b \equiv \frac{4}{3}\pi R_E^3 \quad \text{„effective radius"}$$

$$\therefore \; a = R_E(1-e^2)^{-1/6} = R_E\left(1 + \frac{e^2}{6} + \frac{7}{72}e^4\right)$$

$$\therefore \; b = R_E(1-e^2)^{1/3} = R_E\left(1 - \frac{e^2}{3} - \frac{e^4}{9}\right)$$

* The expansion:

$$\text{Since} \quad 1 = \frac{\rho^2}{a^2} + \frac{z^2}{b^2} = \frac{r^2 s^2\theta}{a^2} + \frac{r^2 c^2\theta}{b^2} = \frac{r^2}{b^2}\left[(1-e^2)s^2\theta + c^2\theta\right]$$

$$\therefore \; T = \frac{b}{\sqrt{1-e^2 s^2\theta}} = R_E(1-e^2)^{1/3}(1-e^2 s^2\theta)^{-1/2} \approx$$

$$\approx R_E\left(1 - \frac{e^2}{3} - \frac{e^4}{9}\right)\left(1 + \frac{1}{2}e^2 s^2\theta + \frac{3}{8}e^4 s^4\theta\right) \eqqcolon$$

$$\eqqcolon \underbrace{R_E\left[1 - \frac{e^2}{3}P_2 + e^4\left(-\frac{P_0}{45} - \frac{11}{63}P_2 + \frac{3}{35}P_4\right)\right] \equiv T_{OS}}_{\text{standard second oblate ellipsoid expansion}}$$

- <u>Matching conditions</u>

* When is T a spheroid? $\Rightarrow T = R(1 + mt + m^2 u)$ describes a topography of an oblate spheroid iff $T_{OS} \equiv \underbrace{T\,(+\text{shift})}_{T'}$

* Since T_{OS} contains P_2 and P_4 only,

$$\boxed{t = t_0 + t_1 P_1 + t_2 P_2} \quad \boxed{u = u_0 + u_1 P_1 + u_2 P_2 + u_3 P_3 + u_4 P_4}$$

* Excentricity expansion : $e^2 = Am + Bm^2$

* Volume constrain :

$$V = \underbrace{V_0}_{\frac{4}{3}\pi R^3}\left(1+3mt_0+3m^2\left(u_0+t_0^2+\tfrac{1}{3}t_1^2+\tfrac{1}{5}t_2^2\right)\right) \equiv \frac{4\pi}{3}R_E^3$$

$$\therefore R = R_E\left(1-mt_0-m^2\left(u_0-t_0^2+\tfrac{1}{3}t_1^2+\tfrac{1}{5}t_2^2\right)\right)$$

* Shift matching : $T \xrightarrow{+\vec{s}=(\alpha m+\beta m^2)\hat{z}R} T' \equiv T_{os}$ no P_1's !

a) $t' = t+\alpha P_1 = t_0+t_1P_1+t_2P_2+\alpha P_1 \therefore \boxed{\alpha = -t_1}$

b) $u' = u+\beta P_1-\alpha\vec{Q}_1\cdot r\nabla t-\frac{1}{3}\alpha^2P_0+\frac{1}{3}\alpha^2P_2 =$

$= u+\beta P_1+t_1\vec{Q}_1\cdot(t_1\vec{Q}_1+t_2\vec{Q}_2)-\frac{1}{3}t_1^2P_0+\frac{1}{3}t_1^2P_2$

$\doteq u_0P_0+u_1P_1+u_2P_2+u_3P_3+u_4P_4+\beta P_1+\frac{6}{5}t_1t_2(P_1-P_3)+$

$+\frac{1}{3}t_1^2-\frac{1}{3}t_1^2P_2 \therefore \boxed{\beta = -u_1-\frac{6}{5}t_1t_2}$

* Final matching

$$T' = R(1+mt'+m^2u') = R_E\left[1-mt_0-m^2\left(u_0-t_0^2+\tfrac{1}{3}t_1^2+\tfrac{1}{5}t_2^2\right)\right]\cdot$$

$$\cdot\left[1+m(t_0P_0+t_2P_2)+m^2\left(u_0P_0+u_2P_2+u_3P_3+u_4P_4-\tfrac{6}{5}t_1t_2P_3+\right.\right.$$

$$\left.\left.+\tfrac{1}{3}t_1^2-\tfrac{1}{3}t_1^2P_2\right)\right] \equiv T_{os}\Big|_{e^2=Am+Bm^2} =$$

$$= R_E\left[1-\frac{m}{3}AP_2+m^2\left(-\frac{1}{3}BP_2+A^2\left(-\frac{P_0}{45}-\frac{11}{63}P_2+\frac{3}{35}P_4\right)\right)\right]$$

$\therefore$ Comparing the coefficients P_2, P_3, P_4 :

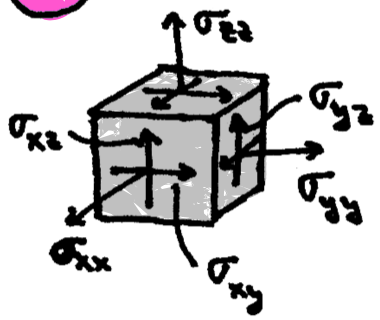
$$\boxed{u_3 = \frac{6}{5}t_1t_2 \; ; \quad u_4 = \frac{27}{35}t_2^2}$$

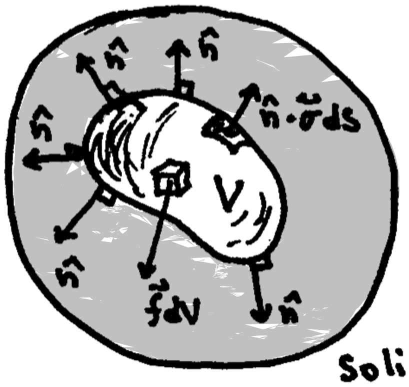
$$\boxed{e^2 = -3t_2m\left[1+\left(\frac{u_2}{t_2}+\frac{11}{7}t_2-t_0-\frac{t_1^2}{3t_2}\right)m\right]}$$

D CONSTITUTIONAL & EQUILIBRIUM BALANCE EQS.

1 MOMENTUM BALANCE

" $\vec{F} = m\vec{a}$ "





Solid, liquid ...

$$\boxed{\rho\vec{a} = \vec{f} + \nabla\cdot\overleftrightarrow{\sigma}}$$

$$\vec{F} = \vec{F}_V + \vec{F}_S = \iiint_V \vec{f}dV + \oiint_S \hat{n}\cdot\overleftrightarrow{\sigma}dS \overset{GT}{=} \iiint_V \vec{f} + \nabla\cdot\overleftrightarrow{\sigma}\, dV$$

* Special cases :

a) Static case $(\vec{a} = 0)$ $\Rightarrow$ $\boxed{\vec{f} + \nabla\cdot\overleftrightarrow{\sigma} = 0}$

b) Potential volume force $\Rightarrow$ $\vec{f} = -\underbrace{\rho}_{\text{density}} \nabla \overbrace{V}^{\text{potential}}$

2 CONSTITUTIONAL EQUATIONS (OVERVIEW)

$\overleftrightarrow{\sigma} = 0$ ... empty space

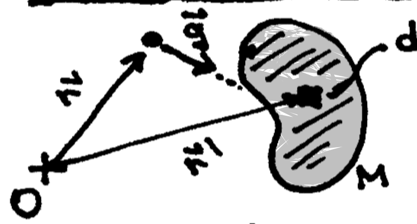
$\overleftrightarrow{\sigma} = -pI$ ... perfect fluid

$\overleftrightarrow{\sigma} = -pI + \eta(\nabla\vec{v} + \nabla\vec{v}^T)$ ... newtonian fluid

$\overleftrightarrow{\sigma} = 2\mu\overleftrightarrow{\varepsilon} + \lambda I \,\mathrm{Tr}\,\overleftrightarrow{\varepsilon}$ ... elastic solid

3 NEWTONIAN GRAVITY

• <u>Acceleration & potential</u>



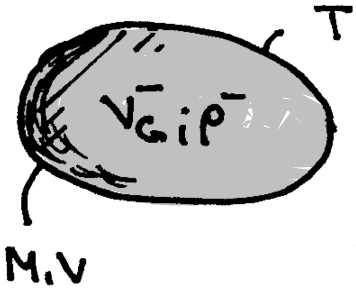
$dM(\vec{r}') = \rho(\vec{r}')dV(\vec{r}')$

* Since $\nabla\frac{1}{r} = -\frac{1}{r^2}\hat{r} = -\frac{\vec{r}}{r^3}$

" $g = \frac{MG}{R^2}$ " $\Rightarrow$ $\vec{g}(r) = G\int_{V'} \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3}\rho(\vec{r}')dV' = G\int_{V'} \nabla \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}dV'$

$\therefore$ $\vec{g} = -\nabla V_G$, where $V_G = -G\int_{V'} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}dV'$

- **Topography induced potential**



$$T = R(1+mt+m^2u) \; ; \; M = \frac{4}{3}\pi R^3 \rho^-$$

$$t = \sum_{n=0}^{\infty} t_n P_n(\hat{r}\cdot\hat{z})$$

$$V_G^+ ; \rho^+ = 0 \qquad u = \sum_{n=0}^{\infty} u_n P_n(\hat{r}\cdot\hat{z})$$

* External potential

$$V_G^+ = -G\int_{r'<T'} \frac{\rho^-(\vec{r}')dV'}{|\vec{r}-\vec{r}'|} \overset{CE}{=} -G\rho^- \sum_{n=0}^{\infty} \oint d\Omega' \int_0^{T'} dr'\, r'^2 \frac{1}{r}\left(\frac{r'}{r}\right)^n P_n(\hat{r}\cdot\hat{r}')$$

$$= -G\rho^- \oint d\Omega' \sum_{n=0}^{\infty} \frac{1}{3+n} \frac{T'^{3+n}}{r^{1+n}} P_n(\hat{r}\cdot\hat{r}') =$$

$$= -G\rho^- R^2 \oint d\Omega' \sum_{n=0}^{\infty} \frac{1}{3+n}\left(\frac{R}{r}\right)^{1+n} (1+mt'+m^2u')^{3+n} P_n(\hat{r}\cdot\hat{r}')$$

$$= -G\rho^- R^2 \oint d\Omega' \sum_{n=0}^{\infty} \frac{1}{3+n}\left(\frac{R}{r}\right)^{1+n}\left(1+(3+n)mt' + (3+n)m^2u' + \frac{1}{2}(3+n)(2+n)t'^2\right) P_n(\hat{r}\cdot\hat{r}')$$

$$= -G\rho^- R^2 \left[\frac{4\pi}{3}\left(\frac{R}{r}\right) + \sum_{n=0}^{\infty} \frac{1}{3+n}\left(\frac{R}{r}\right)^{3+n}(\dots) P_n(\hat{r}\cdot\hat{r}')\right] =$$

$$= -\underbrace{\frac{4\pi G R^2 \rho^-}{3}}_{GM/R}\left[\frac{R}{r} + 3m\sum_{n=0}^{\infty} \frac{t_n P_n}{2n+1}\left(\frac{R}{r}\right)^{1+n} + 3m^2 \sum_{n=0}^{\infty}\frac{u_n P_n}{2n+1}\left(\frac{R}{r}\right)^{1+n} + \frac{3m^2}{8\pi}\sum_{n=0}^{\infty}\oint d\Omega' (2+n)t'^2 P_n(\hat{r}\cdot\hat{r}')\right]$$

* Internal potential

$$V_G^- = -G\int_{r'<T'} \frac{\rho(\vec{r}')dV'}{|\vec{r}-\vec{r}'|} = -G\rho^- \sum_{n=0}^{\infty}\oint d\Omega' \left[\int_0^r dr'\, r'^2 \frac{1}{r}\left(\frac{r'}{r}\right)^n + \int_r^{T'} dr'\, r'^2 \frac{1}{r'}\left(\frac{r}{r'}\right)^n\right] P_n(\hat{r}\cdot\hat{r}')$$

$$= -G\rho^- \oint d\Omega' \left(\frac{r^2}{6} - \frac{r^2}{3} + \sum_{n=0}^{\infty} \frac{1}{2-n}\frac{r^n}{T'^{n-2}}\right) P_n(\hat{r}\cdot\hat{r}') =$$

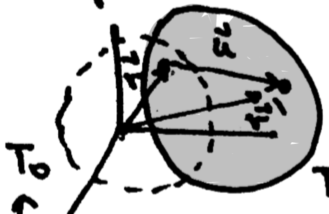
$$= -G\rho^-\left[-\frac{4\pi}{6}r^2 + R^2\sum_{n=0}^{\infty}\frac{1}{2-n}\left(\frac{r}{R}\right)^n \oint d\Omega' P_n(\hat{r}\cdot\hat{r}')\left(1+(2-n)mt' + \frac{1}{2}(2-n)(1-n)m^2t'^2 + (2-n)m^2u'\right)\right] =$$

$$= -\underbrace{\frac{4\pi G R^2\rho^-}{3}}_{GM/R}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2 + 3m\sum_{n=0}^{\infty}\frac{t_n P_n}{2n+1}\left(\frac{r}{R}\right)^2 + 3m^2\sum_{n=0}^{\infty}\frac{u_n P_n}{2n+1}\left(\frac{r}{R}\right)^n + \frac{3m^2}{8\pi}\sum_{n=0}^{\infty}(1-n)\left(\frac{r}{R}\right)^n \oint d\Omega'\, t'^2 P_n(\hat{r}\cdot\hat{r}')\right]$$

4 LINEAR ELASTICITY

- **Deformation**

  * Displacement vector field $\vec{u}$ → describes deformation $\vec{r}' = \vec{r} + \vec{u}(\vec{r})$



  * Topography :

$$T(\hat{r}') = |R\hat{r} + \vec{u}(\vec{r})|\,\big|_R = \sqrt{(R\hat{r}+\vec{u})\cdot(R\hat{r}+\vec{u})}\,\Big|_R = R\sqrt{1+\frac{2\hat{r}\cdot\vec{u}}{R}}\,\Big|_R \approx R + \hat{r}\cdot\vec{u}\,\big|_R$$

and since $\hat{r}' = \hat{r} + O(u)$
and $\vec{u}(\vec{r}) = \vec{u}(\vec{r}') + O(u^2)$ $\therefore$ $\boxed{T(\hat{r}) = R + \hat{r}\cdot\vec{u}\,|_R}$

  * Strain tensor $\overleftrightarrow{\varepsilon}$:

defined by $|d\vec{r}'|^2 - |d\vec{r}|^2 = d\vec{r}\cdot 2\overleftrightarrow{\varepsilon}\cdot d\vec{r}$

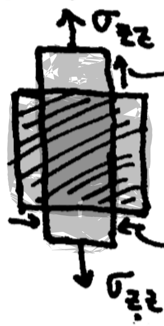
and since $d\vec{r}' = d\vec{r} + d\vec{r}\cdot\nabla\vec{u}$ (Taylor)

$$\Rightarrow |d\vec{r}'|^2 = d\vec{r}'\cdot d\vec{r}' = (d\vec{r} + d\vec{r}\cdot\nabla\vec{u})\cdot(d\vec{r} + \nabla\vec{u}\cdot d\vec{r})$$
$$= d\vec{r}\cdot d\vec{r} + d\vec{r}\cdot(\nabla\vec{u} + \nabla\vec{u}^T)\cdot d\vec{r}$$

$$\therefore \boxed{\overleftrightarrow{\varepsilon} = \frac{1}{2}(\nabla\vec{u} + \nabla\vec{u}^T)}$$

- **Hook's law**

  * Constitutional equation



$\varepsilon_{zz} = \dfrac{\sigma_{zz}}{E}$ → Young's modulus } Material constants
ν → Poisson's ratio

$\varepsilon_{xx} = \varepsilon_{yy} = -\nu\,\varepsilon_{zz}$

Via superposition :

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz})$$
$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz})$$
$$\varepsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy})$$

$$\overleftrightarrow{\varepsilon} = \frac{1+\nu}{E}\overleftrightarrow{\sigma} - \frac{\nu}{E} I \operatorname{Tr}\overleftrightarrow{\sigma} \quad \xrightarrow{\text{Tr}}$$

$$\boxed{\overleftrightarrow{\sigma} = 2\mu\overleftrightarrow{\varepsilon} + \lambda I \operatorname{Tr}\overleftrightarrow{\varepsilon}}$$

$$\mu = \frac{E}{2(1+\nu)}\ ;\ \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \text{Lammé params.}$$

- **Potential decomposition of $\vec{u}$:**

$$\vec{u} = \underbrace{\nabla\phi}_{\text{shift potential}} + \underbrace{\nabla\times\vec{\psi}}_{\text{stream potential}}$$

* Since $\operatorname{Tr}\overleftrightarrow{\varepsilon} = \operatorname{Tr}\frac{1}{2}(\nabla\vec{u}+\nabla\vec{u}^T) = \nabla\cdot\vec{u} = \Delta\phi$

$$\therefore\ \overleftrightarrow{\varepsilon} = \frac{1}{2}(\nabla\vec{u}+\nabla\vec{u}^T) = \nabla^2\phi + \frac{1}{2}(\nabla\nabla\times\vec{\psi} + \nabla\nabla\times\vec{\psi}^T)$$

* $$\overleftrightarrow{\sigma} = 2\mu\overleftrightarrow{\varepsilon} + \lambda I \operatorname{Tr}\overleftrightarrow{\varepsilon} = \mu(\nabla\vec{u}+\nabla\vec{u}^T) + \lambda I \nabla\cdot\vec{u} =$$

$$= 2\mu\nabla^2\phi + \lambda I\Delta\phi + \mu(\nabla\nabla\times\vec{\psi} + \nabla\nabla\times\vec{\psi}^T)$$

* $$\nabla\cdot\overleftrightarrow{\sigma} = \mu\Delta\vec{u} + (\mu+\lambda)\nabla\nabla\cdot\vec{u} = (2\mu+\lambda)\nabla\Delta\phi + \mu\nabla\times\Delta\vec{\psi}$$

- **Free loading & Chree's axisymmetric solution**

$$\vec{f} + \nabla\cdot\overleftrightarrow{\sigma} = 0 \xRightarrow{\vec{f}=0} \nabla\cdot\overleftrightarrow{\sigma} = 0 \quad \text{(free loading)}$$

i.e. $\nabla\cdot\overleftrightarrow{\sigma} = (2\mu+\lambda)\nabla\Delta\phi + \mu\nabla\times\Delta\vec{\psi} = 0$

$$\begin{cases} \phi = \sum_{n=0}^{\infty} \alpha_n r^n P_n + (n+1)\beta_n \mu r^{n+2} P_n \\ \vec{\psi} = \sum_{n=0}^{\infty} (2\mu+\lambda)\beta_n r^{n+2}\vec{R}_n \end{cases}$$

* Justification:

$$\nabla\cdot\overleftrightarrow{\sigma} = (2\mu+\lambda)\nabla\Delta\phi + \mu\nabla\times\Delta\vec{\psi} =$$

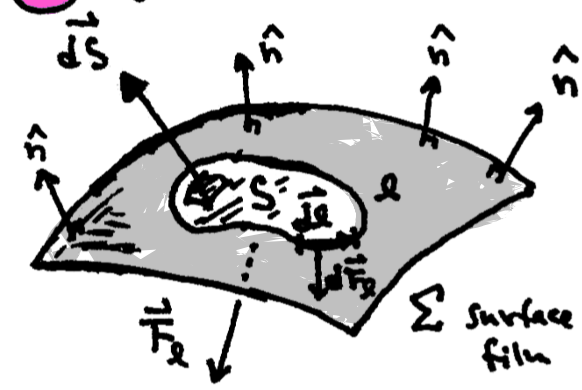
$$= 2\mu(2\mu+\lambda)\sum_{n=0}^{\infty}\beta_n r^n(3+2n)\Big[\underbrace{(n+1)\nabla(r^n P_n) + \nabla\times(r^n\vec{R}_n)}_{0}\Big] = 0$$

where we have used

$$\Delta(r^n P_n) = 0$$

$$\Delta(r^n\vec{R}_n) = \vec{0}$$

⑤ SURFACE TENSION



$\hat n$ ~ unit normal vector field $\perp \Sigma$

model : $dF = \gamma\, d\ell$; $\vec{d\ell} \perp d\vec F \subset \Sigma$

vectorially : $d\vec F = \gamma\, \vec{d\ell} \times \hat n$

surface tension coefficient (material!)

- <u>Total force derivation</u>

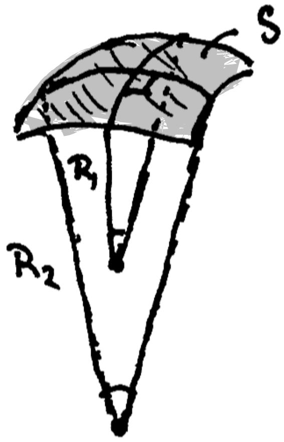
* Recall Stokes thm $\iint_S (\hat n \times \nabla)\, T\, dS = \oint_\ell \vec{d\ell}\, T$

for any tensor T, taking $\times$: $\iint_S \underbrace{(\hat n \times \nabla) \times T}_{\nabla(\hat n \cdot T) - \hat n\, \nabla \cdot T}\, dS = \oint \vec{d\ell} \times T$

* Put $T = \hat n$, then $\nabla(\hat n \cdot T) = \nabla(\hat n \cdot \hat n) = \nabla 1 = \vec 0$, so

$$\vec F_\ell = \oint_\ell d\vec F = \gamma \oint_\ell \vec{d\ell} \times \hat n = -\gamma \iint_S \hat n\, \nabla \cdot \hat n\, dS \approx -\gamma\, \hat n (\nabla \cdot \hat n)\, S$$

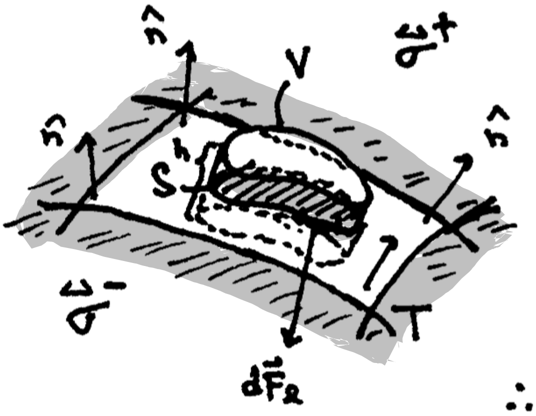
* Note that (Landau-Lifshitz): $\nabla \cdot \hat n = \frac{1}{R_1} + \frac{1}{R_2}$



$$\therefore \quad \vec F_\ell = -\gamma\, \hat n \underbrace{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}_{\substack{\text{mean curvature} \\ \text{(invariant !!)}}} S$$

[Young-Laplace equation]

⑥ SURFACE BALANCE



$$\vec 0 = {}_{"}\vec F - m\vec a" = \iiint_V \vec f\, dV + \oint\!\!\oint_S \hat n \cdot \overleftrightarrow{\sigma}\, dS +$$

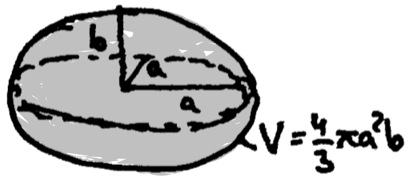
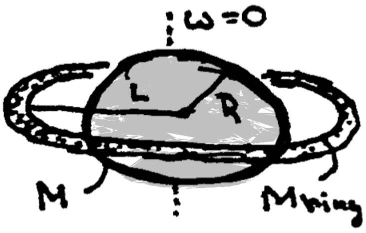
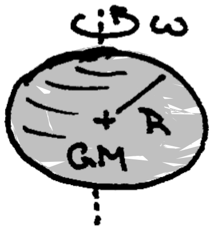
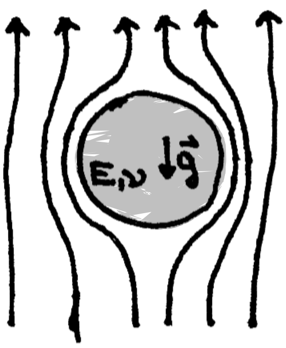
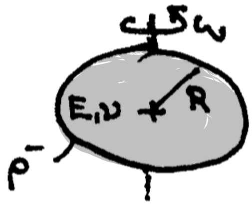
$$+ \oint d\vec F_\ell - \rho V \vec a = \langle h \to 0 \rangle$$

$$\approx \hat n \cdot \overleftrightarrow{\sigma}^+ S - \hat n \cdot \overleftrightarrow{\sigma}^- S - \gamma\, \hat n\, \nabla \cdot \hat n\, S \big|_T$$

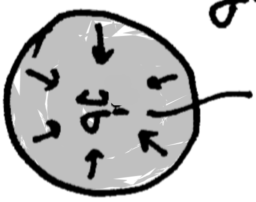
$$\therefore \quad \boxed{\hat n \cdot (\overleftrightarrow{\sigma}^+ - \overleftrightarrow{\sigma}^-) - \gamma\, \hat n\, \nabla \cdot \hat n \big|_T = \vec 0}$$

E WORKED EXAMPLES

TABLE OF HANDY RESULT

Oblate spheroid	Excentricity	Stretch factor
	$e^2 = 1 - \frac{b^2}{a^2}$	$\delta = \frac{V}{V_0} - 1$
	0	$-\frac{3GM\bar{\rho}(1-2\nu)}{2E}$
	$\frac{15 M_{ring}}{4M}\left(\frac{R}{L}\right)^3$	0
	$\frac{5\omega^2 R^3}{2GM}$	0
	$\frac{3M\omega^2}{16\pi\gamma}$	0
	0	0
	$\frac{2R^2\bar{\rho}\omega^2(1+\nu)(2+\nu)}{E(7+5\nu)}$	$\frac{2R^2\bar{\rho}\omega^2(1-2\nu)}{5E}$

1 STATIC SELF-GRAVITATING ELASTIC PLANET



$$\overset{\leftrightarrow}{\sigma}^+ = 0\ ; \quad \vec{f}^+ = 0$$

$$\overset{\leftrightarrow}{\sigma}^- = 2\mu\overset{\leftrightarrow}{\varepsilon} + \lambda I \,\mathrm{Tr}\,\overset{\leftrightarrow}{\varepsilon} \qquad \frac{\vec{u} = \nabla\phi}{\overset{\leftrightarrow}{\varepsilon} = \nabla^2\phi} \qquad 2\mu\nabla^2\phi + \lambda I \Delta\phi$$

$$\vec{f}^- = -\rho^- \nabla \bar{V}_G\ ; \text{ where } \bar{V}_G = -\frac{GM}{R}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right]$$

- Internal ballance

$$\vec{f}^- + \nabla\cdot\overset{\leftrightarrow}{\sigma}^- = \vec{0}\ , \quad \text{but } \nabla\cdot\overset{\leftrightarrow}{\sigma}^- = (2\mu+\lambda)\nabla\Delta\phi$$

$$\therefore \text{ sol } \quad (2\mu+\lambda)\Delta\phi - \rho^-\bar{V}_G = \text{const.}$$

$*$ Ansatz $\quad \phi = Ar^4 + Br^2$

a) $\nabla\phi = 4Ar^3\hat{r} + 2Br\hat{r} \longrightarrow \hat{r}\cdot\nabla\phi = 4Ar^3 + 2Br$

b) $\Delta\phi = 20Ar^2 + 6B$

$\therefore$ Comparing r^2: $(2\mu+\lambda)20A - \rho^-\frac{GM}{2R^3} \equiv 0 \Rightarrow A = \frac{GM\rho^-}{40R^3(2\mu+\lambda)}$

- Surface ballance

$$\hat{n}\cdot(\overset{\leftrightarrow}{\sigma}^+ - \overset{\leftrightarrow}{\sigma}^-)|_T = 0 \longrightarrow \hat{r}\cdot\overset{\leftrightarrow}{\sigma}^-|_R = 0$$

i.e. $\hat{r}\cdot\overset{\leftrightarrow}{\sigma}^-|_R = 2\mu\hat{r}\cdot\nabla^2\phi + \lambda\hat{r}\cdot I\Delta\phi =$

$$= \left[2\mu(12AR^2 + 2B) + \lambda(20AR^2 + 6B)\right]\hat{r} =$$

$$= 2\left[2AR^2(6\mu + 5\lambda) + B(2\mu + 3\lambda)\right]\hat{r} \equiv 0$$

$$\therefore B = -2AR^2\,\frac{6\mu + 5\lambda}{2\mu + 3\lambda}$$

- In total:

$$\phi = \frac{GMR\rho^-}{40(2\mu+\lambda)}\left[\left(\frac{r}{R}\right)^4 - 2\,\frac{6\mu+5\lambda}{2\mu+3\lambda}\left(\frac{r}{R}\right)^2\right]$$

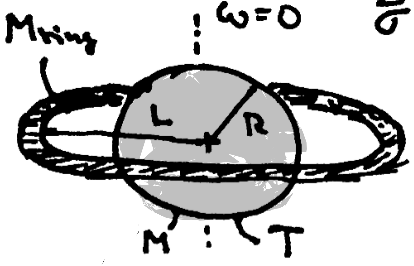
$$\vec{u} = \nabla\phi = \frac{GM\rho^-}{10(2\mu+\lambda)}\left[\left(\frac{r}{R}\right)^3 - \frac{6\mu+5\lambda}{2\mu+3\lambda}\,\frac{r}{R}\right]$$

$$\therefore \vec{u}|_R = -\frac{GM\rho^-\hat{r}}{5(2\mu+3\lambda)} = -\frac{GM\rho^-(1-2\nu)}{5E}\hat{r}$$

$$\text{so } \quad \delta = \frac{V}{V_0} - 1 = \left(\frac{R_E}{R}\right)^3 - 1 = -\frac{3MG\rho^-(1-2\nu)}{2E}$$

2

PLANET TOPOGRAPHY INFLUENCED BY A RING



$\overleftrightarrow{\sigma}^+ = 0$ (empty), $\overleftrightarrow{\sigma}^- = -pI$ (perfect fluid)

$\vec{f}^+ = 0$, $\vec{f}^- = -\rho^- \nabla V^-$, where

$V^- = V_G^- + V_{ring}^-$; $\kappa := \frac{M_{ring}}{2\pi L}$

self gravity — ring-induced grav. potential

$m := \frac{M_{ring}}{M}\left(\frac{R}{L}\right)^3$

a) $V_G^- = -\frac{GM}{R}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2 + 3m \sum \frac{t_n P_n}{2n+1}\left(\frac{r}{R}\right)^n\right]$

b) $V_{ring}^- = -G\oint_{\ell'} \frac{\kappa\, d\ell'}{|\vec{r} - L\hat{\rho}'|} \overset{L \gg R}{=} -G\kappa \int_0^{2\pi} 1 + \frac{r}{L}P_1(\hat{r}\cdot\hat{\rho}') + \frac{r^2}{L^2}P_2(\hat{r}\cdot\hat{\rho}')\,d\varphi'$

$\overset{\varphi OG}{=} -2\pi G\kappa\left[1 + \frac{r}{L}P_1(\hat{r}\cdot\hat{z})P_1(\underbrace{\hat{z}\cdot\hat{\rho}'}_{0}) + \frac{r^2}{L^2}P_2(\hat{r}\cdot\hat{z})P_2(\underbrace{\hat{z}\cdot\hat{\rho}'}_{0})\right]$

$= \pi G\kappa \frac{r^2}{L^2}P_2(\hat{r}\cdot\hat{z}) = \frac{GM}{R}\frac{m}{2}P_2$ (up to a constant)

$\therefore V^- = -\frac{GM}{R}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{m}{2}\left(\frac{r}{R}\right)^2 P_2 + 3m\sum \frac{t_n P_n}{2n+1}\left(\frac{r}{R}\right)^n\right]$

- <u>Internal balance</u>

$\vec{f}^- + \nabla\cdot\overleftrightarrow{\sigma}^- = 0$; but $\nabla\cdot\overleftrightarrow{\sigma}^- = -\nabla p^-$ and $\vec{f}^- = -\overset{\text{constant}}{\rho^-}\nabla V^-$

↪ $-\rho^-\nabla V^- - \nabla p^- = 0$ $\therefore$ solution $\rho^- V^- + p^- = p_0$

- <u>Surface balance</u>

$\hat{n}\cdot(\overleftrightarrow{\sigma}^+ - \overleftrightarrow{\sigma}^-)|_T = 0 \Rightarrow 0 + p^-\hat{n}|_T = 0 \Rightarrow p^-|_T = 0,$

i.e. const. $= \frac{p_0}{\rho^-} = V^-|_T = -\frac{GM}{R}\left[\frac{3}{2} - \frac{1}{2}(1+mt) - \frac{m}{2}P_2 + 3m\sum_{n=0} \frac{t_n P_n}{2n+1}\right]$

$= -\frac{GM}{R}\left[1 - mt - \frac{m}{2}P_2 + 3m\sum_{n=0}^{\infty}\frac{t_n P_n}{2n+1}\right]$

* By comparison of P_n: a) $t_0 = 0$ (fixed volume) $\Rightarrow \frac{p_0}{\rho^-} = -\frac{GM}{R}$

b) $t_1 \equiv 0$ (shift degeneracy); c) $t_2 = -\frac{5}{4}$; d) $t_n = 0 : n \geq 3$

- <u>In total</u>: $T = R(1 - \frac{5}{4}mP_2) \Rightarrow e^2 = -3mt_2 = \frac{15}{4}m = \frac{15 M_{ring}}{4M}\left(\frac{R}{L}\right)^3$

* $V^- = -\frac{GM}{R}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{5}{4}mP_2\left(\frac{r}{R}\right)^2\right] \therefore p = \rho^-\frac{GM}{R}\left[\frac{1}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{5}{4}mP_2\left(\frac{r}{R}\right)^2\right]$

* $\vec{a}^- = -\nabla V^- = \frac{GM}{R}\left[-\frac{r}{R^2}\hat{r} - \frac{5}{4}m(\nabla P_2)\left(\frac{r}{R}\right)^2 - \frac{5}{4}mP_2\frac{r}{R^2}\hat{r}\right] = \frac{GM}{R^2}\left[-\hat{r} - \frac{5}{4}m\vec{Q}_2 - \frac{5}{4}m\vec{P}_2\right]\frac{r}{R}$

↪ $\vec{a}|_T = \frac{GM}{R^2}[\cdots](1 - \frac{5}{4}mP_2) = \frac{GM}{R^2}\left[\hat{r} - \frac{5}{4}m\vec{Q}_2\right]$

planet	♃	♄	⛢	♆
e	10^{-7}	10^{-4}	10^{-5}	10^{-5}

③ ROTATING PLANET UNDER SELF-GRAVITY

$T = R(1 + mt + m^2 u)$; $\overset{\leftrightarrow}{\sigma}^+ = 0$ (empty space);

$\overset{\leftrightarrow}{\sigma}^- = -p^- I$ (perfect fluid), $\rho^- = \text{const.}$ (incompress.);

$V^- = V_G^- + V_{rot}^-$ (in a co-rotating frame)

$m := \dfrac{\omega^2 R^3}{GM}$

a) $\vec{a}_{rot} = \dfrac{v^2}{\rho}\hat{\rho} = \omega^2 \rho \hat{\rho} = \nabla \frac{1}{2}\omega^2 \rho^2 \equiv -\nabla V_{rot}^-$

$\therefore V_{rot}^- = -\frac{1}{2}\omega^2\rho^2 = -\frac{R^2\omega^2}{2}\left(\frac{r}{R}\right)^2 s^2\theta = -\frac{GM}{R}\frac{m}{3}\left(\frac{r}{R}\right)^2(P_0 - P_2)$

b) $\therefore V^- = -\frac{GM}{R}\Big[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2 + \frac{m}{3}\left(\frac{r}{R}\right)^2(P_0 - P_2) + 3m\sum_{n=0}^{\infty}\frac{t_n P_n}{2n+1}\left(\frac{r}{R}\right)^2 +$

$+ 3m^2\sum_{n=0}^{\infty}\frac{u_n P_n}{2n+1}\left(\frac{r}{R}\right)^2 + \frac{3m^2}{8\pi}\sum_{n=0}^{\infty}(1-n)\left(\frac{r}{R}\right)^n \oint t'^2 P_n(\hat{r}\cdot\hat{r}')\,d\Omega'\Big]$

- <u>Internal balance</u>

$\vec{f}^- + \nabla\cdot\overset{\leftrightarrow}{\sigma}^- = 0 \Rightarrow -\rho^-\nabla V^- - \nabla p^- = 0$ $\therefore$ sol $\rho^- V^- + p^- = p_0^-$ (const.)

- <u>Surface balance</u>

$\hat{n}\cdot(\overset{\leftrightarrow}{\sigma}^+ - \overset{\leftrightarrow}{\sigma}^-)|_T = 0 \Rightarrow p^-|_T = 0 \quad \therefore \quad \dfrac{p_0^-}{\rho^-} = V^-|_T$

* First order:

$\dfrac{p_0^-}{\rho^-} = V^-|_T \overset{O(m)}{=} -\dfrac{GM}{R}\Big[1 - mt + \frac{m}{3}(P_0 - P_2) + 3m\sum_{n=0}^{\infty}\frac{t_n P_n}{2n+1}\Big]$

a) $t_0 = 0$ (vol. fix.) ; b) $t_1 \equiv 0$ (shift degeneracy)

c) $t_2 = -\frac{5}{6}$; d) $t_n = 0$; $n \geq 3$ $\quad \therefore \; t = -\frac{5}{6}P_2$

* Second order:

Recall $P_2^2 = \frac{1}{5}P_0 + \frac{2}{7}P_2 + \frac{18}{35}P_4$, hence

$V^- = -\frac{GM}{R}\Big[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2 + \frac{m}{3}\left(\frac{r}{R}\right)^2(P_0 - P_2) - \frac{m}{2}\left(\frac{r}{R}\right)^2 P_2 +$

$+ 3m^2\sum_{n=0}^{\infty}\frac{u_n P_n}{2n+1}\left(\frac{r}{R}\right)^n + \frac{3m^2}{2}t_2^2\left(\frac{1}{5}P_0 - \frac{2}{7}\frac{1}{5}\left(\frac{r}{R}\right)^2 P_2 - 3\cdot\frac{18}{35}\frac{1}{9}\left(\frac{r}{R}\right)^2 P_4\right)\Big]$

$= -\frac{GM}{R}\Big[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2 + \frac{m}{3}\left(\frac{r}{R}\right)^2 - \frac{5m}{6}\left(\frac{r}{R}\right)^2 P_2 + \frac{5m^2}{24} - \frac{5}{84}m^2\left(\frac{r}{R}\right)^2 P_2 -$

$- \frac{5}{28}m^2\left(\frac{r}{R}\right)^4 P_4 + 3m^2\sum_{n=0}^{\infty}\frac{u_n P_n}{2n+1}\left(\frac{r}{R}\right)^n\Big]$

$\therefore \dfrac{p_0^-}{\rho^-} = V^-|_T \overset{O(m^2)}{=} -\dfrac{GM}{R}\Big[\frac{3}{2} - \frac{1}{2}(1 + mt + m^2u)^2 + \frac{m}{3}(1 + mt)^2 - \frac{5m}{6}(1+mt)^2 P_2 +$

$+ \frac{5m^2}{24} - \frac{5}{84}m^2 P_2 - \frac{5}{28}m^2 P_4 + 3m^2\sum_{n=0}^{\infty}\frac{u_n P_n}{2n+1}\left(\frac{r}{R}\right)^n\Big] =$

$$= -\frac{GM}{R}\left[1+\frac{m}{3}+m^2\left(-u-\frac{t^2}{2}+\frac{2}{3}t-\frac{5}{3}tP_2+\frac{5}{24}-\frac{5}{84}P_2-\frac{5}{28}P_4+3\sum_{n=0}^{\infty}\frac{u_nP_n}{2n+1}\right)\right] =$$

$$= -\frac{GM}{R}\left[1+\frac{m}{3}+m^2\left(-u+\frac{5}{24}-\frac{155}{252}P_2-\frac{5}{28}P_4+\frac{25}{24}P_2^2+3\sum_{n=0}^{\infty}\frac{u_nP_n}{2n+1}\right)\right] =$$

$$= -\frac{GM}{R}\left[1+\frac{m}{3}+m^2\left(\frac{5}{12}-\frac{20}{63}P_2+\frac{5}{14}P_4-u+3\sum_{n=0}^{\infty}\frac{u_nP_n}{2n+1}\right)\right]$$

* By comparison of coefficients
  a) (vol. fix.): $u_0=-\frac{1}{5}t_0^2=-\frac{5}{36}$ ∴ $\frac{p_0^-}{\rho^-}=-\frac{GM}{R}\left[1+\frac{m}{3}+\frac{5m^2}{36}\right]$
  b) $u_1=0$ (shift degeneracy)
  c) $u_2=-\frac{50}{63}$; d) $u_3=0$; e) $u_4=\frac{15}{28}$; f) $u_n=0: n\geq 5$
  ∴ $u=-\frac{5}{36}-\frac{50}{63}P_2+\frac{15}{28}$

• <u>In total</u>:

$$T=R\left[1-\frac{5}{6}mP_2+m^2\left(-\frac{5}{36}-\frac{50}{63}P_2+\frac{15}{28}P_4\right)\right]$$

* Ellipsoid matching conditions:
  a) $\frac{6}{5}t_1t_2=\frac{6}{5}\cdot 0\cdot\left(-\frac{5}{6}\right)=0\equiv u_3$ ✓
  b) $\frac{27}{35}t_2^2=\frac{27}{35}\left(-\frac{5}{6}\right)^2=\frac{15}{28}\equiv u_4$ ✓
  ∴ $e^2=-3t_2m\left[1+\left(\frac{u_2}{t_2}+\frac{11}{7}t_2\right)m\right]\overset{\cdots}{=}\frac{5}{2}m\left(1-\frac{5}{14}m\right)$
  $\Rightarrow$ OR $m=\frac{2}{5}e^2\left(1+\frac{e^2}{7}\right)$

[NOTE: <u>exact</u> $m=\frac{1}{2e^3}\left[9e(e^2-1)-3\sqrt{1-e^2}\,(2e^2-3)\arcsin e\right]$

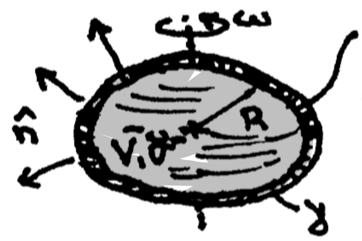
• <u>Internal variables</u>

* $V^-=-\frac{GM}{R}\left[\frac{3}{2}-\frac{1}{2}\left(\frac{r}{R}\right)^2+\frac{m}{3}\left(\frac{r}{R}\right)^2-\frac{5m}{6}\left(\frac{r}{R}\right)^2P_2-\frac{5}{24}m^2-\frac{15}{28}m^2\left(\frac{r}{R}\right)^2P_2\right]$
* $p^-=-\rho^-\frac{GM}{R}\left[\frac{1}{2}-\frac{1}{2}\left(\frac{r}{R}\right)^2-\frac{m}{3}+\frac{m}{3}\left(\frac{r}{R}\right)^2-\frac{5m}{6}\left(\frac{r}{R}\right)^2P_2-\frac{25}{72}m^2-\frac{15}{28}m^2\left(\frac{r}{R}\right)^2P_2\right]$
* $\vec{a}^-=-\nabla V^-\overset{\cdots}{=}\frac{GM}{R^2}\frac{r}{R}\left[-\hat{r}+\frac{2}{3}m\hat{r}-\frac{5}{3}m\vec{P}_2-\frac{5m}{6}\vec{Q}_2-\frac{15}{14}m^2\vec{P}_2-\frac{15}{28}m^2\vec{Q}_2\right]$
* $\vec{a}^-|_T=\frac{GM}{R^2}\left[-\hat{r}+\frac{2}{3}m\hat{r}-\frac{5}{6}m(\vec{P}_2+\vec{Q}_2)+\frac{5}{12}m^2\hat{r}-\frac{55}{126}m^2(\vec{P}_2+\vec{Q}_2)+\frac{5}{28}m^2(\vec{P}_4+\vec{Q}_4)\right]$

?	☉	☿	♀	⊕	☾	♂	♃	♄	♅	♆
e	0.0068	0.0011	0.0004	0.093	0.0044	0.11	0.45	0.57	0.27	0.25

predicted eccentricities

4 ROTATING DROPLET UNDER SURFACE TENSION



$T = R(1 + mt + m^2 u)$; $\overleftrightarrow{\sigma}^- = -p^- I$ (perf. fluid)

$V = \frac{4}{3}\pi R^3$ $\qquad \overleftrightarrow{\sigma}^+ = 0$ (empty space)

$M = V\rho^-$; $\rho^- =$ const.

$$m := \frac{\rho^- R^3 \omega^2}{12\gamma} \qquad V^- = V^-_{rot} = -\frac{1}{2}\omega^2\rho^2 = -\frac{6\gamma m}{\rho^- R}\left(\frac{r}{R}\right)^2 \sin^2\theta =$$

$$= -\frac{4\gamma}{\rho^- R} m \left(\frac{r}{R}\right)^2 (P_0 - P_2) \quad \text{(co-rotating r.f.)}$$

$$\hat{n} = \hat{r} - mR\nabla t - \frac{m^2}{2}R^2\hat{r}\,|\nabla t|^2 - m^2 R\nabla u$$

- <u>Internal balance</u>

$f^- + \nabla\cdot\overleftrightarrow{\sigma}^- = 0$, i.e. $\rho^-\nabla V^- - \nabla p^- = 0 \therefore$ sol $\rho^- V^- + p^- = p_0^-$ (const.)

- <u>Surface balance</u>

$\hat{n}\cdot(\overleftrightarrow{\sigma}^+ - \overleftrightarrow{\sigma}^-) - \gamma\hat{n}\nabla\cdot\hat{n}|_T = 0 \Rightarrow \hat{n}p^- - \gamma\hat{n}\nabla\cdot\hat{n}|_T = 0,$

so $p^- - \gamma\nabla\cdot\hat{n}|_T = 0 \therefore$ sol const. $= p_0^- = \gamma\nabla\cdot\hat{n} + \rho^- V^-|_T$

* First order:

a) $\hat{n} = \hat{r} - mR\nabla t + O(m^2)$

$$\nabla\cdot\hat{n} = \nabla\cdot\hat{r} - mR\Delta t = \frac{2}{r} - mR\sum_{n=0}^{\infty} t_n \Delta P_n = \frac{2}{r} + \frac{mR}{r^2}\sum_{n=0}^{\infty} t_n n(n+1)P_n$$

$$\nabla\cdot\hat{n}|_T = \frac{2}{R(1+mt)} + \frac{m}{R}\sum_{n=0}^{\infty} n(n+1)t_n P_n = \frac{2}{R} - \frac{2m}{R}t + \frac{m}{R}\sum_{n=0}^{\infty} n(n+1)t_n P_n$$

b) $\rho^- V^-|_T = -\frac{4\gamma}{R} m\,(P_0 - P_2)$

c) $\therefore p_0^- = \gamma\nabla\cdot\hat{n} + \rho^- V^-|_T = \frac{\gamma}{R}\left[2 - 2mt - 4m(P_0 - P_2) + m\sum_{n=0}^{\infty} n(n+1)t_n P_n\right]$

Comparing: $t_0 = 0$ (vol. fix); $t_1 \equiv 0$ (shift fix.); $t_2 = -1$

so $t = -P_2 \Rightarrow T = R(1 - mP_2) \therefore e^2 = -3t_2 m = \frac{\rho^- R^3\omega^2}{4\gamma} = \frac{3M\omega^2}{16\pi\gamma}$

* Second order:

a) $|\nabla t|^2 = \frac{1}{r}\vec{Q}_2\cdot\vec{Q}_2 = \frac{1}{r^2}\left(\frac{6}{5}P_0 + \frac{6}{7}P_2 - \frac{72}{35}P_4\right)$

$\therefore \hat{n} = \hat{r} - mR\nabla t - m^2\left(\frac{R}{r}\right)^2\hat{r}\left(\frac{3}{5}P_0 + \frac{3}{7}P_2 - \frac{36}{35}P_4\right) - m^2 R\nabla u$

Since $\nabla\cdot\frac{\hat{r}}{r^2} = \frac{2}{r}\frac{1}{r^2} - \frac{2}{r^3}\hat{r}\cdot\hat{r} = 0 \quad (r \neq 0)$

$$\therefore \nabla\cdot\hat{n} = \frac{2}{r} - mR\Delta t - m^2 R\Delta u = \frac{2}{r} - \frac{6mR}{r^2}P_2 + \frac{m^2 R}{r^2}\sum_{n=0}^{\infty} n(n+1)u_n P_n$$

$$\therefore \nabla\cdot\hat{n}\big|_T = \frac{2}{R}(1+mt+m^2u)^{-1} - \frac{6}{R}m(1+mt)^{-2}P_2 + \frac{m^2}{R}\sum_{n=0}^{\infty} n(n+1)u_nP_n =$$

$$= \frac{1}{R}\left[2-2mt-2m^2u+2m^2t^2-6mP_2+12m^2tP_2+m^2\sum\cdots\right]$$

$$= \frac{1}{R}\left[2-4mP_2-2m^2u-10m^2P_2^{\,2}+m^2\sum_{n=0}^{\infty}n(n+1)u_nP_n\right]$$

b) $\bar{V}\big|_T = -\frac{4\gamma m}{\rho^- R}(1+mt)^2(P_0-P_2) = -\frac{4\gamma}{\rho^- R}\left[m-mP_2+2m^2t-2m^2tP_2\right]$

c) $\therefore\ \bar{p}_0 = \gamma\,\nabla\cdot\vec{n} + \rho^-\ \bar{V}\big|_T = \frac{\gamma}{R}\left[2-4m+8m^2P_2-18m^2P_2^{\,2}-2m^2u+m^2\sum\cdots\right]$

$$= \frac{\gamma}{R}\left[2-4m-\frac{18}{5}m^2+\frac{20}{7}m^2P_2-\frac{324}{35}m^2P_4-2m^2u+m^2\sum_{n=0}^{\infty}n(n+1)u_nP_n\right]$$

Comparing:

$u_0 = -\frac{1}{5}t_2^{\,2} = -\frac{1}{5}$ ($\because$ vol. fix.) $\Rightarrow \bar{p}_0 = \frac{\gamma}{R}\left[2-4m-\frac{16}{5}m^2\right]$

$u_1 \equiv 0$ (shift degeneracy)

$u_2 = -5/7$; $u_3 = 0$; $u_4 = \frac{18}{35}$; $u_n = 0 \quad n\geq 5$

so $u = -\frac{1}{5} - \frac{5}{7}P_2 + \frac{18}{35}P_4$

- <u>In total</u>

* Topography $T = R\left[1-mP_2+m^2\left(-\frac{1}{5}P_0-\frac{5}{7}P_2+\frac{18}{35}P_4\right)\right]$

* Pressure $\bar{p} = \frac{\gamma}{R}\left[2-4m-\frac{16}{5}m^2+4m\left(\frac{r}{R}\right)^2(P_0-P_2)\right]$

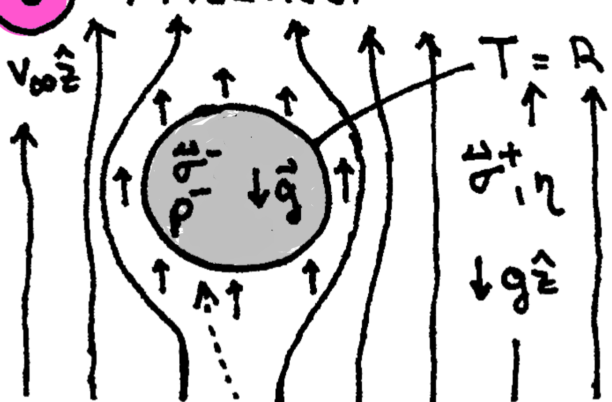
* Oblate spheroid matching condition?

a) $\frac{6}{5}t_1t_2 = \frac{6}{5}\cdot 0\cdot(-1) = 0 \equiv u_3$ ✓

b) $\frac{27}{35}t_2^{\,2} = \frac{27}{35} \neq \frac{18}{35} \equiv u_4$ ✗

NOT an oblate spheroid!

5 FALLING ELASTIC BALL



$T = R + \hat r\cdot\vec u|_R$; $\overleftrightarrow{\sigma}^- = 2\mu\overleftrightarrow{\varepsilon} + \lambda I\,\mathrm{Tr}\,\overleftrightarrow{\varepsilon}$

$\hat r\cdot\overleftrightarrow{\sigma}^+|_R = \frac{3V_\infty\eta}{2R}\hat z$ (Stokes flow)

$\left(\vec F = \oint \hat n\cdot\overleftrightarrow{\sigma}^+ dS = 6\pi R V_\infty\eta\hat z\right)$

~ co~~f~~-falling frame

undeformed state :

$g=0$; $T_0=R$; $V_0=\frac{4}{3}\pi R^3$;

Global balance: $Mg = 6\pi R V_\infty\eta$ ∴ $\hat r\cdot\overleftrightarrow{\sigma}^+|_R = \frac{1}{3}R\rho^- g\hat z$

- Internal balance :

$\vec f^- + \nabla\cdot\overleftrightarrow{\sigma}^- = \vec 0$; $\vec f^- = -\rho^- g\hat z = -\nabla(\rho^- g z)$

* Displacement vector decomposition: $\vec u = \nabla\phi + \nabla\times\vec\psi$

a) $\overleftrightarrow{\sigma}^- = 2\mu\nabla^2\phi + \lambda I\Delta\phi + \mu(\nabla\nabla\times\vec\psi + \nabla\nabla\times\vec\psi^T)$

b) $\nabla\cdot\overleftrightarrow{\sigma}^- = (2\mu+\lambda)\nabla\Delta\phi + \mu\,\nabla\times\Delta\vec\psi \equiv -\vec f = \nabla(\rho^- g z)$

* Split of solutions: $\phi = \phi_p + \phi^*$; $\vec\psi = \vec\psi_p + \vec\psi^*$

∴ $\vec u = \vec u_p + \vec u^*$; $\overleftrightarrow{\sigma}^- = \overleftrightarrow{\sigma}_p + \overleftrightarrow{\sigma}^*$; ...

* Particular (p) solution : We can choose $\vec\psi_p = 0$ and ϕ_p such that it satisfies $(2\mu+\lambda)\Delta\phi_p = \rho^- g z$

a) Ansatz $\tilde\phi_p = \alpha z^3 \Rightarrow \Delta\tilde\phi_p = 6\alpha z$, comparing: $\boxed{\alpha = \frac{\rho^- g}{6(2\mu+\lambda)}}$

b) However $\tilde\phi_p = \alpha z^3 = \alpha r^3 c^3\theta = \alpha r^3 P_1^3 = \alpha r^3\left(\frac{3}{5}P_1 + \frac{2}{5}P_3\right)$,

and since $\Delta r^3 P_3 \Rightarrow \phi_p = \frac{3}{5}\alpha r^3 P_1$

c) Note that $z^3 = r^3\left(\frac{3}{5}P_1 + \frac{2}{5}P_3\right)$, so taking Δ:

$6z = \frac{3}{5}\Delta(r^3 P_1)$, from which $\Delta(r^3 P_1) = 10 r P_1$.

d) Also $\vec r\times\nabla z^3 = \vec r\times\nabla\left(r^3\left(\frac{3}{5}P_1 + \frac{2}{5}P_3\right)\right) = r^3\left(\frac{3}{5}\vec R_1 + \frac{2}{5}\vec R_3\right)/\Delta$

$\Delta(\vec r\times\nabla z^3) = \vec r\times\nabla\Delta z^3 = \vec r\times\nabla 6z = 6r\vec R_1 \equiv \frac{3}{5}\Delta(r^3\vec R_1)$ ∴ $\Delta(r^3\vec R_1) = 10 r\vec R_1$

* Internal balance is therefore achieved when (*): $\underbrace{\nabla\cdot\overleftrightarrow{\sigma}^* = 0}_{\text{free loading}}$

∴ Gen. sol. $\phi^* = \sum_n \alpha_n r^n P_n + \mu\beta_n r^{n+2} P_n$

[Chree] $\vec\psi^* = \sum_n (2\mu+\lambda)\beta_n r^{n+2}\vec R_n$

- <u>Surface balance</u>:

$$\hat{n}\cdot(\overleftrightarrow{\sigma}^{+}-\overleftrightarrow{\sigma}^{-})|_T=0 \overset{O(\vec{u})}{\Longrightarrow} \hat{r}\cdot(\overleftrightarrow{\sigma}^{+}-\overleftrightarrow{\sigma}_p-\overleftrightarrow{\sigma}^{*})|_R=0$$

$\therefore\ \hat{r}\cdot\overleftrightarrow{\sigma}^{*}|_R=\hat{r}\cdot\overleftrightarrow{\sigma}^{+}|_R-\hat{r}\cdot\overleftrightarrow{\sigma}_p|_R$ (boundary condition on $\overleftrightarrow{\sigma}^{*}$)

* $\overleftrightarrow{\sigma}_p=2\mu\nabla^2\phi_p+\lambda I\Delta\phi_p$, so

a) $\nabla\phi_p=\frac{3}{5}\alpha r^2(3\vec{P}_1+\vec{Q}_1)$

b) $\hat{r}\cdot\nabla^2\phi_p=\partial_r\nabla\phi_p=\frac{6}{5}\alpha r(3\vec{P}_1+\vec{Q}_1)$

c) $\therefore\ \hat{r}\cdot\overleftrightarrow{\sigma}_p|_R=\frac{12}{5}\alpha\mu R(3\vec{P}_1+\vec{Q}_1)+6\lambda\alpha R\vec{P}_1$

* Since $\hat{z}=\nabla z=\nabla(rP_1)=\vec{P}_1+\vec{Q}_1$

$\therefore\ \hat{r}\cdot\overleftrightarrow{\sigma}^{+}|_R=\frac{1}{3}R\bar{\rho}g\,\hat{z}=2R\alpha(2\mu+\lambda)(\vec{P}_1+\vec{Q}_1)$

* Overall $\hat{r}\cdot\overleftrightarrow{\sigma}^{*}|_R\overset{\cdots}{=}\frac{2}{5}R\alpha\,(4\mu+5\lambda)(\vec{Q}_1-2\vec{P}_1)$

- <u>Internal balance again</u>

* $\overleftrightarrow{\sigma}^{*}=2\mu\nabla^2\phi^{*}+\lambda I\Delta\phi^{*}+\mu(\nabla\nabla\times\vec{\psi}^{*}+\nabla\nabla\times\vec{\psi}^{*T})$

a) based on BC's, we choose $\phi^{*}=2A\mu r^3P_1$; $\vec{\psi}^{*}=A(2\mu+\lambda)r^3\vec{R}_1$

b) $\nabla\phi^{*}=2A\mu r^2(3\vec{P}_1+\vec{Q}_1)\ \therefore\ \hat{r}\cdot\nabla^2\phi^{*}|_R=4A\mu R(3\vec{P}_1+\vec{Q}_1)$

and $\Delta\phi^{*}|_R=20A\mu RP_1$

c) $\nabla\times\vec{\psi}^{*}=2A(2\mu+\lambda)r^2(-2\vec{Q}_1-\vec{P}_1)$

$\therefore\ \hat{r}\cdot\nabla\nabla\times\vec{\psi}^{*}|_R=4A(2\mu+\lambda)R(-2\vec{Q}_1-\vec{P}_1)$

and $\hat{r}\cdot(\nabla\nabla\times\vec{\psi}^{*})^T|_R=2A(2\mu+\lambda)(\vec{Q}_1-2\vec{P}_1)$

* Overall $\hat{r}\cdot\overleftrightarrow{\sigma}^{*}|_R\overset{\cdots}{=}2A\mu(2\mu+3\lambda)R(2\vec{P}_1-\vec{Q}_1)\equiv\hat{r}\cdot(\overleftrightarrow{\sigma}^{+}-\overleftrightarrow{\sigma}_p)|_R$

$\therefore$ Comparing, we get $A=-\dfrac{\alpha}{5}\dfrac{4\mu+5\lambda}{\mu(2\mu+3\lambda)}=-\dfrac{\bar{\rho}g(4\mu+5\lambda)}{30\mu(2\mu+\lambda)(2\mu+3\lambda)}$

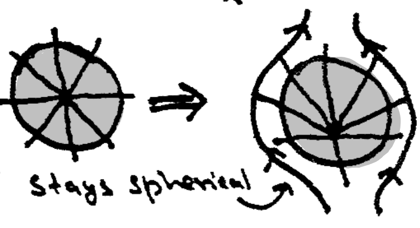
- <u>In total</u>:

* $\vec{u}=\nabla\phi_p+\nabla\phi^{*}+\nabla\times\vec{\psi}^{*}\overset{\cdots}{=}\dfrac{\bar{\rho}gr^2}{6\mu(2\mu+3\lambda)}\left[(\mu+2\lambda)\vec{P}_1+(3\mu+4\lambda)\vec{Q}_1\right]$

* Note that if $\vec{u}$ is a solution, then $\vec{u}+d\hat{z}=\vec{u}+d(\vec{P}_1+\vec{Q}_1)$ is <u>also</u> a sol. ($\hat{z}$-shift) $\therefore\ \vec{u}=\dfrac{\bar{\rho}gR^2}{6\mu(2\mu+3\lambda)}\left[(\mu+2\lambda)\left(\frac{r^2}{R^2}-1\right)\vec{P}_1+\left(\frac{r^2}{R^2}(3\mu+4\lambda)-(\mu+2\lambda)\right)\vec{Q}_1\right]$

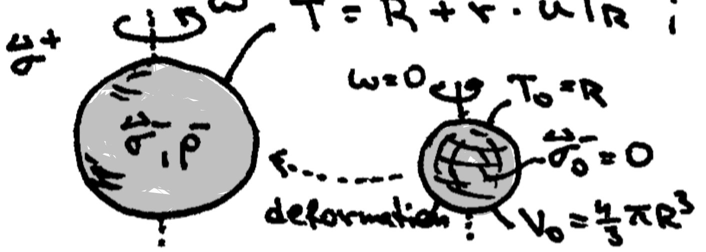
a) $\therefore\ \vec{u}|_R=\dfrac{\bar{\rho}gR^2(\mu+\lambda)}{3\mu(2\mu+3\lambda)}\vec{Q}_1=\dfrac{\bar{\rho}gR^2(1-2\nu)}{3E}\vec{Q}_1\ \therefore\ T=R+\hat{r}\cdot\vec{u}|_R=R$ (?)

b) $\vec{u}|_{r=0}=-\dfrac{\bar{\rho}gR^2(\mu+2\lambda)}{6\mu(2\mu+3\lambda)}\hat{z}=-\dfrac{\bar{\rho}gR^2(1+2\nu)}{6E}\hat{z}$



c) Stored energy: $E_{ela}=\int_V\frac{1}{2}\overleftrightarrow{\sigma}:\overleftrightarrow{\varepsilon}\,dV\overset{\cdots}{=}\dfrac{3\pi Rv_{\infty}^2\eta^2(10\lambda+7\mu)}{10\mu(3\lambda+2\mu)}$

6 ROTATING ELASTIC BALL



$T = R + \hat{r}\cdot\vec{u}|_R$; $\overleftrightarrow{\sigma}^- = 2\mu\overleftrightarrow{\varepsilon} + \lambda I \,\mathrm{Tr}\,\overleftrightarrow{\varepsilon}$;

$\overleftrightarrow{\sigma}^+ = 0$ (empty space) ;

$\bar{V} = \bar{V}_{rot} = -\frac{1}{2}\omega^2\rho^2$.

- **Internal balance:**

$\vec{f}^- + \nabla\cdot\overleftrightarrow{\sigma}^- = \vec{0}$; $\vec{f} = -\rho^-\nabla\bar{V} = \nabla\left(\frac{1}{2}\omega^2\rho^2\rho^-\right)$

* Displacement decomposition $\vec{u} = \nabla\phi + \nabla\times\vec{\psi}$, so again

a) $\overleftrightarrow{\sigma}^- = 2\mu\nabla^2\phi + \lambda I\Delta\phi + \mu\left(\nabla\nabla\times\vec{\psi} + \nabla\nabla\times\vec{\psi}^T\right)$

b) $\nabla\cdot\overleftrightarrow{\sigma}^- = (2\mu+\lambda)\nabla\Delta\phi + \mu\nabla\times\Delta\vec{\psi} \equiv -\vec{f} = -\nabla\left(\frac{1}{2}\omega^2\rho^2\rho^-\right)$

* Split of the solution: $\phi = \phi_p + \phi^*$; $\vec{\psi} = \vec{\psi}_p + \vec{\psi}^*$; $\vec{u} = \dots, \overleftrightarrow{\sigma} = \dots$

* Particular (p) solution: $\vec{\psi}_p = 0$ & ϕ_p so $(2\mu+\lambda)\Delta\phi_p = -\frac{1}{2}\omega^2\rho^2\rho^-$

a) Ansatz $\tilde{\phi}_p = \alpha\rho^4 \Rightarrow \nabla\tilde{\phi} = 4\alpha\rho^3\hat{\rho}$, taking $\nabla\cdot$:

$\Delta\tilde{\phi}_p = \nabla\cdot\nabla\tilde{\phi}_p = 4\alpha\left(3\rho^2\hat{\rho}\cdot\hat{\rho} + \rho^3\underbrace{\nabla\cdot\hat{\rho}}_{1/\rho}\right) = 16\alpha\rho^2$;

Comparing: $\boxed{\alpha = -\dfrac{\rho^-\omega^2}{32(2\mu+\lambda)}}$

b) However $\tilde{\phi}_p = \alpha\rho^4 = \alpha r^4 s^4\theta = \alpha r^4\left(\frac{8}{15}P_0 - \frac{16}{21}P_2 + \frac{8}{35}P_4\right)$

and since $\Delta(r^4P_4) = 0 \Rightarrow \phi_p = \alpha r^4\left(\frac{8}{15}P_0 - \frac{16}{21}P_2\right)$

c) Note that taking Δ of $\rho^4 = r^4\left(\frac{8}{15}P_0 - \frac{16}{21}P_2 + \frac{8}{35}P_4\right)$, we get

$16\rho^2 = \frac{8}{15}5\cdot4r^2 - \frac{16}{21}\Delta(r^4P_2) \therefore \Delta(r^4P_2) = 14r^2P_2$

d) Also since $\vec{r}\times\nabla\rho^4 = \vec{r}\times\nabla\left(r^4\left(\frac{8}{15} - \frac{16}{21}P_2 + \frac{8}{35}P_4\right)\right) = r^4\left(-\frac{16}{21}\vec{R}_2 + \frac{8}{35}\vec{R}_4\right)$

but $\Delta(\vec{r}\times\nabla\rho^4) = \vec{r}\times\nabla\Delta\rho^4 = 16\vec{r}\times\nabla\rho^2 = 16\vec{r}\times\nabla\left(r^2\left(\frac{2}{3}(P_0 - P_2)\right)\right) = -\frac{2}{3}r^2 16\vec{R}_2$

$\equiv \Delta\left(r^4\left(-\frac{16}{21}\vec{R}_2 + \frac{8}{35}\vec{R}_4\right)\right) = -\frac{16}{21}\Delta(r^4\vec{R}_2) \therefore \Delta(r^4\vec{R}_2) = 14r^2\vec{R}_2$

* Internal balance is then achieved when $\nabla\cdot\overleftrightarrow{\sigma}^* = 0 \Rightarrow$ [Chree's sol.]

- **Surface balance**

$\hat{n}\cdot\left(\overleftrightarrow{\sigma}^+ - \overleftrightarrow{\sigma}^-\right)|_T = 0 \overset{O(\vec{u})}{\Longrightarrow} \hat{r}\cdot\overleftrightarrow{\sigma}^*|_R = -\hat{r}\cdot\overleftrightarrow{\sigma}_p|_R$ (BC for $\overleftrightarrow{\sigma}^*$)

* $\overleftrightarrow{\sigma}_p = 2\mu\nabla^2\phi_p + \lambda I\Delta\phi_p$, so

a) $\nabla\phi_p = \nabla\, 8\alpha r^4\left(\frac{1}{15}P_0 - \frac{2}{21}P_2\right) = 8\alpha r^3\left(\frac{4}{15}\vec{P}_0 - \frac{8}{21}\vec{P}_2 - \frac{2}{21}\vec{Q}_2\right)$

b) $\hat{r}\cdot\nabla^2\phi_p = \partial_r\nabla\phi_p = 24\alpha r^2\left(\frac{4}{15}\vec{P}_0 - \frac{8}{21}\vec{P}_2 - \frac{2}{21}\vec{Q}_2\right)$

c) and since $\Delta\phi_p = 16\alpha\rho^2 = 16\alpha r^2 s^2\theta = \frac{32}{3}\alpha r^2(P_0 - P_2)$, we get

$*\quad \therefore \hat{r}\cdot\overleftrightarrow{\sigma}_p|_R = 96\alpha\mu R^2\left(\frac{2}{15}\vec{P}_0 - \frac{4}{21}\vec{P}_2 - \frac{1}{21}\vec{Q}_2\right) + \frac{32\alpha}{3}\lambda R^2(\vec{P}_0 - \vec{P}_2)$

$*$ Overall $\hat{r}\cdot\overleftrightarrow{\sigma}^*|_R = -\hat{r}\cdot\overleftrightarrow{\sigma}_p|_R = 32\alpha R^2\left[-\frac{2\mu}{5}\vec{P}_0 + \frac{4\mu}{7}\vec{P}_2 + \frac{\mu}{7}\vec{Q}_2 - \frac{\lambda}{3}\vec{P}_0 + \frac{\lambda}{3}\vec{P}_2\right]$

- <u>Internal balance again</u>

$*\quad \overleftrightarrow{\sigma}^* = 2\mu\nabla^2\phi^* + \lambda I\Delta\phi^* + \mu(\nabla\nabla\times\vec{\psi}^* + \nabla\nabla\times\vec{\psi}^{*T})$

a) Based on BC's, we choose (Chee's sol. ansatz):

$$\phi^* = A\mu r^2 P_0 + Br^2 P_2 + 3C\mu r^4 P_2$$

$$\vec{\psi}^* = \underbrace{\cancel{A(2\mu+\lambda)\vec{R}_0}}_{0} + C(2\mu+\lambda)r^4\vec{R}_2$$

b) $\nabla\phi^* = 2A\mu r\vec{P}_0 + Br(2\vec{P}_2 + \vec{Q}_2) + 3C\mu r^3(4\vec{P}_2 + \vec{Q}_2)$

$\therefore\ \hat{r}\cdot\nabla^2\phi^*|_R = 2A\mu\vec{P}_0 + B(2\vec{P}_2 + \vec{Q}_2) + 9C\mu R^2(4\vec{P}_2 + \vec{Q}_2)$

and $\Delta\phi^*|_R = 6A\mu P_0 + 42C\mu R^2 P_2$

c) $\nabla\times\vec{\psi}^* = C(2\mu+\lambda)r^3(-5\vec{Q}_2 - 6\vec{P}_2)$

$\therefore\ \hat{r}\cdot\nabla\nabla\times\vec{\psi}^*|_R = \partial_r\nabla\times\vec{\psi}|_R = 3C(2\mu+\lambda)R^2(-5\vec{Q}_2 - 6\vec{P}_2)$

and $\hat{r}\cdot(\nabla\nabla\times\vec{\psi}^*)^T|_R = (\nabla\nabla\times\vec{\psi}^*)\cdot\hat{r}|_R \overset{...}{=} C(2\mu+\lambda)R^2(-18\vec{P}_2 - \vec{Q}_2)$;

so adding these two: $\hat{r}\cdot(\nabla\nabla\times\vec{\psi}^* + \nabla\nabla\times\vec{\psi}^{*T})|_R = C(2\mu+\lambda)R^2(-36\vec{P}_2 - 16\vec{Q}_2)$

$*$ Overall $\hat{r}\cdot\overleftrightarrow{\sigma}^*|_R = \hat{r}\cdot[2\mu\nabla^2\phi^* + \lambda I\Delta\phi^* + \mu(\nabla\nabla\times\vec{\psi}^* + \nabla\nabla\times\vec{\psi}^{*T})]|_R \overset{...}{=}$

$= 2\mu[2A\mu\vec{P}_0 + B(2\vec{P}_2 + \vec{Q}_2) + 9C\mu R^2(4\vec{P}_2 + \vec{Q}_2)] + \lambda[6A\mu\vec{P}_0 + 42C\mu R^2\vec{P}_2] +$

$+ C\mu(2\mu+\lambda)R^2(-36\vec{P}_2 - 16\vec{Q}_2) \overset{BC}{\equiv} 32\alpha R^2\left[-\frac{2\mu}{5}\vec{P}_0 + \frac{4\mu}{7}\vec{P}_2 + \frac{\mu}{7}\vec{Q}_2 - \frac{\lambda}{3}\vec{P}_0 + \frac{\lambda}{3}\vec{P}_2\right]$

$\therefore$ Comparing, we get (solving linear system of eqs.):

$$A = -\frac{16\alpha R^2(6\mu+5\lambda)}{15\mu(2\mu+3\lambda)};\ B = \frac{32\alpha R^2(2\mu+\lambda)(3\mu+4\lambda)}{3\mu(14\mu+19\lambda)};\ C = \frac{16\alpha(6\mu+7\lambda)}{21\mu(14\mu+19\lambda)}$$

- <u>In total</u>:

$*\quad \vec{u} = \nabla\phi_p + \nabla\phi^* + \nabla\times\vec{\psi}^* = 16\alpha r^3\left(\frac{2}{15}\vec{P}_0 - \frac{4}{21}\vec{P}_2 - \frac{1}{21}\vec{Q}_2\right) + 2A\mu r\vec{P}_0 +$

$+ Br(2\vec{P}_2 + \vec{Q}_2) + 3C\mu r^3(4\vec{P}_2 + \vec{Q}_2) + C(2\mu+\lambda)r^3(-5\vec{Q}_2 - 6\vec{P}_2)$

a) $\vec{u}|_R \overset{...}{=} -\frac{64\alpha R^3(2\mu+\lambda)}{15(2\mu+3\lambda)} + \frac{32\alpha R^3(2\mu+\lambda)(4\mu+5\lambda)}{3\mu(14\mu+19\lambda)}\vec{P}_2 + \frac{16\alpha R^3(2\mu+\lambda)(2\mu+3\lambda)}{3\mu(14\mu+19\lambda)}\vec{Q}_2$

b) $\therefore T = R + \hat{r}\cdot\vec{u}|_R \overset{...}{=} R + R^3\bar{\rho}\omega^2\left[\frac{2}{15(2\mu+3\lambda)} - \frac{4\mu+5\lambda}{3\mu(14\mu+19\lambda)}P_2\right] \equiv$

$\equiv T_{OS} = R_E(1 - \frac{1}{3}e^2 P_2) = R(1 + \frac{1}{3}\delta - \frac{1}{3}e^2 P_2)$ (oblate spheroid topography)

$\therefore$ Comparing: $\delta = \frac{V}{V_0} - 1 = \frac{2R^2\bar{\rho}\omega^2}{5(2\mu+3\lambda)} = \frac{2R^2\bar{\rho}\omega^2(1-2\nu)}{5E}$; $e^2 = R^2\bar{\rho}\omega^2\frac{4\mu+5\lambda}{\mu(14\mu+19\lambda)} = \frac{2R^2\bar{\rho}\omega^2(1+\nu)(2+\nu)}{E(7+5\nu)}$