A new perspective on the standard models of viscoelastic fluids (W.I.P.)

Michal Bathory

Faculty of Mathematics and Physics
Charles University

Xi'an
January 16 2025

For an unknown divergence-free velocity $\mathbf{v} : Q \to \mathbb{R}^d$, pressure $p : Q \to \mathbb{R}$ and extra stress tensor $\mathbf{B} : Q \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, consider the system

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \operatorname{div}(\mu\,\mathbf{B}), \quad \nu,\mu > 0,$$

$$\partial_t \mathbf{B} + \mathbf{v}\cdot\nabla\mathbf{B} + \frac{1}{\tau}(\mathbf{B}^\alpha - \mathbf{B}^{\alpha-1}) = (\nabla\mathbf{v})\,\mathbf{B} + \mathbf{B}\,(\nabla\mathbf{v})^T, \quad \tau > 0,\ \alpha \geq 1. \tag{B}$$

- $\alpha = 1$ is the Oldroyd-B model (1950) and $\alpha = 2$ is the Giesekus model (1962).
- Equation (B) can be written as

$$\overset{\triangledown}{\mathbf{B}} + \frac{1}{\tau}(\mathbf{B}^\alpha - \mathbf{B}^{\alpha-1}) = 0,$$

where

$$\overset{\triangledown}{\mathbf{B}} := \partial_t\mathbf{B} + \mathbf{v}\cdot\nabla\mathbf{B} - (\nabla\mathbf{v})\,\mathbf{B} - \mathbf{B}\,(\nabla\mathbf{v})^T$$

$$= \underbrace{\partial_t\mathbf{B} + \mathbf{v}\cdot\nabla\mathbf{B} + \mathbf{B}\,W - W\,\mathbf{B}}_{\overset{\circ}{\mathbf{B}}\ \text{Jaumann-Zaremba}} - (\mathbf{B}\,D + D\,\mathbf{B}) = \overset{\circ}{\mathbf{B}} - (\mathbf{B}\,D + D\,\mathbf{B})$$

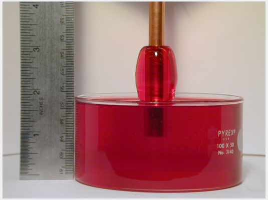
is the upper convected (Oldroyd) derivative and

$$D := \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T), \qquad W := \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^T).$$

(Dis)advantages of the Oldroyd-B/Giesekus models

+ There are compelling arguments for considering equation (B) with $\overset{\triangledown}{\mathbf{B}}$:

- $\overset{\triangledown}{\mathbf{B}}$ is an objective derivative (rate), i.e. $\overset{\triangledown}{\mathbf{B}}^*(t,x^*) = Q(t)\overset{\triangledown}{\mathbf{B}}(t,x)Q(t)^T$ if $\mathbf{B}^*(t,x^*) = Q(t)\mathbf{B}(t,x)Q(t)^T$ for any (non-stationary) rotation $Q(t)$ of the observer.
- When $\mathbf{B} = \boldsymbol{F}\boldsymbol{F}^T$, $\boldsymbol{F} = \partial_{\boldsymbol{X}}\chi(t,\boldsymbol{X})$ (solid mechanics) then $\overset{\triangledown}{\mathbf{B}} = 0$.
- $\overset{\triangledown}{\mathbf{B}}$ arises by downscaling certain microscopic models.
- Established in applications (polymeric fluids, rod climbing etc.).
- The Giesekus case $\alpha = 2$ (remarkably) admits a three-dimensional global weak solution for any initial data due to the recent result (Los et al. 2024).



− However, there are also drawbacks:

- $\overset{\triangledown}{\mathbf{B}}$ is just one of really **many** objective derivatives.
- $\overset{\triangledown}{\mathbf{B}}$ is **not corotational**. Cor. derivatives are **superior** in physics, analysis and numerics.
- In the Oldroyd-B case $\alpha = 1$, the apriori estimates are not sufficient for $(\nabla\mathbf{v})\mathbf{B} + \mathbf{B}(\nabla\mathbf{v})^T$.
- The equation is on the verge of being "ill-posed", comparing just with $u' + u^\alpha = gu$, $g \in L^2$.

The equivalence result

If done **properly**, the idea of multiplying the equation (B) by $\mathbf{B}^{-1}$ leads to the following:

Theorem

Let $\boldsymbol{v} : Q \to \mathbb{R}^d$, $\boldsymbol{H} : Q \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *be smooth and set*

$$\mathbf{B} = e^{2\boldsymbol{H}}, \quad \text{so that} \quad \boldsymbol{H} = \frac{1}{2}\log\mathbf{B} \qquad \text{is the logarithmic (Hencky) strain.}$$

Then $\mathbf{B}$ *solves the Oldroyd-B/Giesekus equation* (B) *if and only if* $\boldsymbol{H}$ *solves*

$$\boxed{\partial_t \boldsymbol{H} + \boldsymbol{v}\cdot\nabla\boldsymbol{H} + \frac{1}{2\tau}\left(e^{2(\alpha-1)\boldsymbol{H}} - e^{2(\alpha-2)\boldsymbol{H}}\right) = \boldsymbol{D} - \boldsymbol{H}\boldsymbol{\Omega}^{\log} + \boldsymbol{\Omega}^{\log}\boldsymbol{H},} \tag{H}$$

where:

$$\boldsymbol{\Omega}^{\log} := \boldsymbol{W} - \sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D} \qquad \text{is the logaritmic spin,}$$

$$\mathrm{ad}_{\boldsymbol{H}} X := \boldsymbol{H}X - X\boldsymbol{H}, \quad X \in \mathbb{R}^{d\times d}, \qquad \text{is the commutator,}$$

$$\sigma(x) := \coth x - \frac{1}{x}, \quad x \in \mathbb{R}. \qquad \text{is an odd function.}$$

Defining $\sigma(\mathrm{ad}_{\boldsymbol{H}})$

- The power series approach is problematic since coth has poles on the imaginary axis, and so the formal power series of $\sigma(\mathrm{ad}_{\boldsymbol{H}})$ may not converge.
- Instead, one can proceed more directly, using (pointwise) the Schur diagonalization

$$\boldsymbol{h} = Q^T \boldsymbol{H} Q, \quad \boldsymbol{h} = \mathrm{diag}\,(h_i)_{i=1}^d, \quad Q^{-1} = Q^T.$$

- Then, for any function $f : \mathbb{R} \to \mathbb{R}$ we can put

$$f(\mathrm{ad}_{\boldsymbol{H}})X := Q\big((f(h_i - h_j))_{ij} \odot (Q^T X Q)\big) Q^T, \quad X \in \mathbb{R}^{d\times d}.$$

- This resembles the Daleckĭ-Kreĭn formula, but is not quite. In fact, the differences $h_i - h_j$ arise here as the eigenvalues of $\mathrm{ad}_{\boldsymbol{H}}$.
- One can verify that this definition is in alignment with the expected calculus (e.g. if f is is holomorphic), in particular, that it is independent of the choice of Q. Moreover, there holds

$$f(\mathrm{ad}_{\boldsymbol{H}})g(\mathrm{ad}_{\boldsymbol{H}}) = g(\mathrm{ad}_{\boldsymbol{H}})f(\mathrm{ad}_{\boldsymbol{H}}) = (fg)(\mathrm{ad}_{\boldsymbol{H}})$$

which is a key property in subsequent computations.

Explicit formulas

One may object that we still do not know **how to calculate** $\sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D}$. To this end, thanks to the Caley-Hamilton theorem, we can prove the following **explicit** representation formulas:

One may object that we still do not know **how to calculate** $\sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D}$. To this end, thanks to the Caley-Hamilton theorem, we can prove the following **explicit** representation formulas:

Lemma

$$\boxed{\sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D} = \frac{\sigma\left(\sqrt{\mathrm{tr}^2\,\boldsymbol{H} - 4\det\boldsymbol{H}}\right)}{\sqrt{\mathrm{tr}^2\,\boldsymbol{H} - 4\det\boldsymbol{H}}}(\boldsymbol{HD} - \boldsymbol{DH}) \quad \textit{if } d = 2,}$$

and

$$\boxed{\sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D} = -P_2(\boldsymbol{HD} - \boldsymbol{DH}) + P_1(\boldsymbol{H}^2\boldsymbol{D} - \boldsymbol{DH}^2) - P_0(\boldsymbol{H}^2\boldsymbol{DH} - \boldsymbol{HDH}^2) \quad \textit{if } d = 3,}$$

where P_n are invariants of $\boldsymbol{H}$ defined by

$$P_n := \frac{h_1^n\sigma(h_2 - h_3) + h_2^n\sigma(h_3 - h_1) + h_3^n\sigma(h_1 - h_2)}{(h_1 - h_2)(h_2 - h_3)(h_3 - h_1)}, \quad n = 0, 1, 2.$$

In fact, this holds analogously for any (odd) function. One can also note already here that $\sigma(\mathrm{ad}_{\boldsymbol{H}})$ is a **bounded continuous** function of $\boldsymbol{H}$.

Properties of the model (H)

The obtained logarithmic model has a number of remarkable properties.

$$\boxed{\begin{aligned}\partial_t \boldsymbol{H} + \mathbf{v}\cdot\nabla\boldsymbol{H} + \frac{1}{2\tau}\left(e^{2(\alpha-1)\boldsymbol{H}} - e^{2(\alpha-2)\boldsymbol{H}}\right) = \boldsymbol{D} - \boldsymbol{H}\,\boldsymbol{\Omega}^{\log} + \boldsymbol{\Omega}^{\log}\boldsymbol{H},\\ \boldsymbol{\Omega}^{\log} = W - \sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D}.\end{aligned}} \tag{H}$$

The logarithmic derivative (*log*-rate)

$$\overset{\circ}{\boldsymbol{H}}{}^{\log} := \partial_t \boldsymbol{H} + \mathbf{v}\cdot\nabla\boldsymbol{H} + \boldsymbol{H}\,\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\boldsymbol{H} \tag{1}$$

is objective and **corotational**. It is interesting to compare our approach with the theory of Xiao et al. (1998) for corotational derivatives (in context of the finite elastoplasticity), see

Xiao, H.; Bruhns, O. T. & Meyers, A. T. M.: *Strain rates and material spins*, Journal of Elasticity, 1998, 52, 1-41.

They argue that **every** objective corotational derivative arises through (1), where the spin tensor is

$$\boldsymbol{\Omega} = \boldsymbol{W} + \boldsymbol{\Gamma}(\mathbf{B}, \boldsymbol{D}) \quad \text{for some isotropic antisymmetric function } \boldsymbol{\Gamma}.$$

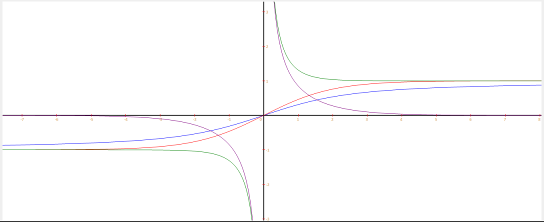
Xiao et al. further conclude that the spin tensor should be determined by a single **spin function** $\tilde{h}$ and provide the representation in eigenprojections $\mathbf{B}_i$ of the form

$$\boldsymbol{\Omega} = \boldsymbol{W} + \sum_{i \neq j}^{d} \tilde{h}\left(\frac{b_i}{b_j}\right)\mathbf{B}_i \boldsymbol{D} \mathbf{B}_j \quad \text{for a continuous } \tilde{h} \text{ satisfying } \tilde{h}(z^{-1}) = -\tilde{h}(z).$$

We note that $\tilde{h}(\frac{b_i}{b_j}) = \tilde{h}(e^{2(h_i - h_j)}) = -f(h_i - h_j)$ for an **odd** function f. Thus, my approach gives an equivalent characterization of these admissible corotational derivatives via

$$\boldsymbol{\Omega} = \boldsymbol{W} - f(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D} \quad \text{for a continuous } f \text{ satisfying } f(-x) = -f(x).$$

Virtually all the spin tensors encountered in practice can be recovered in this way:



0	$\boldsymbol{W}$	Jaumann-Zaremba
$\tanh \frac{x}{2}$	$\boldsymbol{\Omega}^R = \dot{\boldsymbol{R}}\boldsymbol{R}^T$	angular velocity
$\operatorname{csch} x$	$\boldsymbol{\Omega}^L$	Lagrangian twirl
$\coth x$	$\boldsymbol{\Omega}^E$	Eulerian twirl
$\underbrace{\coth x - x^{-1}}_{\sigma(x)}$	$\boldsymbol{\Omega}^{\log}$	Logarithmic spin

Unique position of $\boldsymbol{H}$ and $\Omega^{\log}$

- Within the solid mechanics, the dependence

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H} \approx \boldsymbol{D} \tag{2}$$

  had long been foreseen, but the fact that one has to take **precisely** $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H} \equiv \overset{\circ}{\boldsymbol{H}}{}^{\log}$ was shown only in 1991 (Th. Lehmann, Z.H. Guo and H.Y. Liang).
- In fact, in order to have only $\boldsymbol{D}$ on the right-hand side of (2), one **has to select** the stress measure $\boldsymbol{H}$ and the rate $\overset{\circ}{\boldsymbol{H}}{}^{\log}$ **among all** possible stress measures **and** objective corotational rates. (Xiao et al. 1998)
- This shows that the choice of $\boldsymbol{H} = \frac{1}{2}\log \mathbf{B}$ as the stress measure and the choice of $\overset{\circ}{\boldsymbol{H}}{}^{\log}$ as the objective derivative is quite special.

Back to fluids

- In our **visco**elastic case, there is an additional damping term in the equation

$$\overset{\circ}{\boldsymbol{H}}^{\log} + \frac{1}{2\tau}\left(e^{2(\alpha-1)\boldsymbol{H}} - e^{2(\alpha-2)\boldsymbol{H}}\right) = \boldsymbol{D}$$

- We have basically two ways how to deal with it:
  1) Use the Caley-Hamilton theorem for the matrix exponential: $e^{2\boldsymbol{H}} = \alpha_0\mathbf{I} + \alpha_1\boldsymbol{H} + \alpha_2\boldsymbol{H}^2$, where α_i are exponential-like functions of invariants of $\boldsymbol{H}$. Then deal with such nonlinearities.
  2) **Linearize**.

Back to fluids

- In our **visco**elastic case, there is an additional damping term in the equation

$$\overset{\circ}{\boldsymbol{H}}^{\log} + \frac{1}{2\tau}(e^{2(\alpha-1)\boldsymbol{H}} - e^{2(\alpha-2)\boldsymbol{H}}) = \boldsymbol{D}$$

- We have basically two ways how to deal with it:
  1) Use the Caley-Hamilton theorem for the matrix exponential: $e^{2\boldsymbol{H}} = \alpha_0\mathbf{I} + \alpha_1\boldsymbol{H} + \alpha_2\boldsymbol{H}^2$, where α_i are exponential-like functions of invariants of $\boldsymbol{H}$. Then deal with such nonlinearities.
  2) **Linearize**.
- Linearization merges the Oldroyd-B and Giesekus cases together, since

$$\frac{1}{2\tau}(e^{2(\alpha-1)\boldsymbol{H}} - e^{2(\alpha-1)\boldsymbol{H}}) = \frac{1}{\tau}\boldsymbol{H} + O(|\boldsymbol{H}|^2), \quad \boldsymbol{H} \to 0.$$

- Hence, the linearized model becomes simply

$$\boxed{\overset{\circ}{\boldsymbol{H}}^{\log} + \frac{1}{\tau}\boldsymbol{H} = \boldsymbol{D}}$$

Some arguments in favour of the linearized model

- Recently, Alrashdi & Giusteri (2024) provided a physical derivation of the model:

$$\overset{\triangledown}{\mathbf{B}} + \frac{1}{\tau}\mathbf{B}\log\mathbf{B} = 0.$$

- **Now** we know that its logarithmic counterpart is precisely $\overset{\circ}{\boldsymbol{H}}^{\log} + \frac{1}{\tau}\boldsymbol{H} = \boldsymbol{D}$!
- Note that

$$\mathbf{B}\log\mathbf{B} = \frac{1}{2}(\mathbf{B}-\mathbf{I}) + \frac{1}{2}(\mathbf{B}^2-\mathbf{B}) + O(|\mathbf{B}-\mathbf{I}|^3) \quad \text{as} \quad \mathbf{B}\to\mathbf{I}.$$

- Moreover, Alrashdi & Giusteri provide convincing arguments in favor of including $\boldsymbol{H}$ into the Cauchy stress tensor, instead of $\mathbf{B}$.
- The underlying Helmholtz free energies then are

$$\psi_{\boldsymbol{H}} = \frac{\mu}{2}|\boldsymbol{H}|^2 \quad \text{Linearized model;} \qquad \psi_{\mathbf{B}} = \frac{\mu}{4}\operatorname{tr}(e^{2\boldsymbol{H}} - \mathbf{I} - 2\boldsymbol{H}) \quad \text{Oldroyd-B.}$$

The choice $\psi_{\boldsymbol{H}}$ is somehow favoured and extensively studied in the works by Neff, using more specifically the decomposition

$$2\mu|\operatorname{dev}\boldsymbol{H}|^2 + \kappa|\operatorname{tr}\boldsymbol{H}|^2.$$

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \operatorname{div}(\mu\boldsymbol{H}).$$

$$\partial_t \boldsymbol{H} + \mathbf{v}\cdot\nabla\boldsymbol{H} + \frac{1}{\tau}\boldsymbol{H} - \lambda\Delta\boldsymbol{H} = \boldsymbol{D} + \mathrm{ad}_{\boldsymbol{H}}(\sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D} - \boldsymbol{W}). \tag{3}$$

- The energy identity is

$$\frac{1}{2}\partial_t\int_\Omega(|\mathbf{v}|^2 + \mu|\boldsymbol{H}|^2) + \int_\Omega(\nu|\nabla\mathbf{v}|^2 + \frac{1}{\tau}|\boldsymbol{H}|^2 + \lambda|\nabla\boldsymbol{H}|^2) = 0,$$

  which provides sufficiently strong estimates to define everything in (3).
- Indeed, the only problematic term could be $\sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D}$, but we have $|\sigma| \leq 1$ everywhere, and hence also $\sigma(\mathrm{ad}_{\boldsymbol{H}})$ **is bounded** and continuous w.r.t. to $\boldsymbol{H}$ (thanks to C.H. formulas).
- Moreover, if $\lambda > 0$, then an approximation $\boldsymbol{H}_n$ converges pointwise a.e., and thanks to the explicit formulas, we conclude that the right-hand side of (3) is weakly compact.
- $\lambda = 0$: Arguments by Lions & Masmoudi (2000) show that $\boldsymbol{H}_n \overset{L^2}{\to} \boldsymbol{H}$ if $\sigma = 0$.
  The crucial ingredient in their proof is testing with $\frac{\boldsymbol{H}}{1+\delta|\boldsymbol{H}|^2}$, which **works regardless of** σ.
- Note that the positive definiteness of $\mathbf{B}$ is for granted; it follows from $\mathbf{B} = e^{2\boldsymbol{H}}$.

Proof of (B) $\Leftrightarrow$ (H)

In order to pass from $\partial_t \boldsymbol{H}$ to $\partial_t \mathbf{B} = \partial_t e^{2\boldsymbol{H}}$, for instance, one has to apply the operator $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{D}} = \frac{\mathrm{d}e^{2\boldsymbol{H}}}{\mathrm{d}\boldsymbol{H}} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$, which can be given by the integral (Wilcox 1967):

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}} X = \int_0^1 e^{2(1-s)\boldsymbol{H}} X e^{2s\boldsymbol{H}} \,\mathrm{d}s.$$

Since

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}} X \cdot Y = \int_0^1 e^{(1-s)\boldsymbol{H}} X e^{s\boldsymbol{H}} \cdot e^{(1-s)\boldsymbol{H}} Y e^{s\boldsymbol{H}} \,\mathrm{d}s$$

(here $U \cdot V := \sum_{i,j}^{d} U_{ij} V_{ij}$, $|U| := \sqrt{U \cdot U}$), the operator $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}$ is **symmetric and positive definite**, in particular its inverse $(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}})^{-1}$ exists. This we want to apply to the Oldroyd-B equation, but how to do it explicitly?

Proof of the equivalence, pt. II

There is also an alternative formula, well known in the theory of Lie groups:

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}X = \mathbf{B}\Big(\frac{1 - e^{-2\,\mathrm{ad}_{\boldsymbol{H}}}}{\mathrm{ad}_{\boldsymbol{H}}}X\Big) \quad \text{or} \quad \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}X = \Big(\frac{e^{2\,\mathrm{ad}_{\boldsymbol{H}}} - 1}{\mathrm{ad}_{\boldsymbol{H}}}X\Big)\mathbf{B}.$$

Using our calculus for $f(\mathrm{ad}_{\boldsymbol{H}})$, it is now easy to express the inverse:

$$\Big(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}\Big)^{-1}Y = \frac{\mathrm{ad}_{\boldsymbol{H}}}{1 - e^{-2\,\mathrm{ad}_{\boldsymbol{H}}}}(\mathbf{B}^{-1}Y) \quad \text{or} \quad \Big(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}\Big)^{-1}Y = \frac{\mathrm{ad}_{\boldsymbol{H}}}{e^{2\,\mathrm{ad}_{\boldsymbol{H}}} - 1}(Y\,\mathbf{B}^{-1}).$$

We take advantage of this dichotomy to calculate what is

$$\Big(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}\Big)^{-1}(\mathbf{B}\,\boldsymbol{D} + \boldsymbol{D}\,\mathbf{B}) \quad \text{and} \quad \Big(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}\Big)^{-1}(\mathbf{B}\,\boldsymbol{W} - \boldsymbol{W}\,\mathbf{B}).$$

Proof of the equivalence, pt. III

As a consequence of the elementary identities

$$\frac{x}{1-e^{-x}} - \frac{x}{e^{x}-1} = x \quad \text{and} \quad \frac{x}{1-e^{-x}} + \frac{x}{e^{x}-1} = x \coth\frac{x}{2},$$

and the aforementioned calculus, we easily get

$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}\right)^{-1}(\mathbf{B}\,\boldsymbol{W} - \boldsymbol{W}\,\mathbf{B}) = \mathrm{ad}_{\boldsymbol{H}}\,\boldsymbol{W}, \quad \text{and} \quad \left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}\right)^{-1}(\mathbf{B}\,\boldsymbol{D} + \boldsymbol{D}\,\mathbf{B}) = \left(\mathrm{ad}_{\boldsymbol{H}} \coth \mathrm{ad}_{\boldsymbol{H}}\right)\boldsymbol{D}.$$

Finally, noting that

$$\left(\mathrm{ad}_{\boldsymbol{H}} \coth \mathrm{ad}_{\boldsymbol{H}}\right)\boldsymbol{D} = \boldsymbol{D} + \mathrm{ad}_{\boldsymbol{H}}\,\sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D},$$

we see that indeed

$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\boldsymbol{H}}\right)^{-1}(\nabla\mathbf{v}\,\mathbf{B} + \mathbf{B}(\nabla\mathbf{v})^{T}) = \boldsymbol{D} + \mathrm{ad}_{\boldsymbol{H}}(\boldsymbol{W} + \sigma(\mathrm{ad}_{\boldsymbol{H}})\boldsymbol{D}) = \boldsymbol{D} + \boldsymbol{H}\,\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\boldsymbol{H},$$

and hence we arrive at (H). □