UNIVERSAL ALGEBRA II – HOMEWORK

To get the credit for the tutorial solve either two easy problems, or one hard problem. Easy problems are marked.

1. Arithmetical varieties

Definition. An algebra is *arithmetical* if and only if it is has both distributive congruence lattice and permutable congruences. A variety is arithmetical if every member is.

Problem 1.1 (easy)**.** Prove directly (without using neither Mal'cev, nor Jónsson theorems) that the following are equivalent for a variety $\mathcal{V}$

(1) $\mathcal{V}$ is arithmetical,
(2) $\mathcal{V}$ satisfies the congruence 'identity'

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \circ (\alpha \wedge \gamma),$$

(3) there is a term p of $\mathcal{V}$ (called *Pixley term*) such that

$$\mathcal{V} \models p(x,y,x) \approx p(x,y,y) \approx p(y,y,x) \approx x.$$

Problem 1.2 (easy)**.** Prove that an algebra $\mathbb{A}$ is arithmetical if and only if the following version of Chinese remainer theorem is true.

For all $\theta_1, \dots, \theta_n \in \operatorname{Con} \mathbb{A}$ and all $a_1, \dots, a_n \in A$ such that $a_i \equiv a_j \pmod{\theta_i \vee \theta_j}$ there is $x \in A$ sattisfying for all i

$$x \equiv a_i \pmod{\theta_i}.$$

2. Parallelogram terms imply few subpowers

Definition. A $(k+3)$-ary term t is k-parallelogram term if it satisfies identities

$$t\left(\begin{array}{ccc|cccc} x & y & y & z & x & \dots & x \\ y & y & x & x & z & \dots & x \\ & \vdots & & & & \ddots & \\ y & y & x & x & x & \dots & z \end{array}\right) = \begin{pmatrix} x \\ x \\ \vdots \\ x \end{pmatrix}.$$

(The vertical line is for a better readability.)

Problem 2.1. Prove that if an algebra has a parallelogram term (of some arity) then it has few subpowers. (Hint. Combine proofs of Mal'cev and near-unanimity cases.)

Date: May 20, 2015.

3. Term rewriting systems

Problem 3.1 (easy)**.** Prove that the following rewrite system is convergent.

$$s(x) + (y + z) \approx x + (ss(y) + z) \tag{1}$$

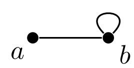
$$s(x) + (y + (z + w)) \approx x + (z + (y + w)) \tag{2}$$

(The signature is $(+,\, s,\, 0)$, the symbols are binary, unary, and nulary, respectively.)

Problem 3.2. For the rewriting system from the previous problem, consider the function f which assigns to n the length of the longest derivation on input with n symbols. Show that f cannot be bounded from above by any primitive recursive function. (Hint: Consider the function $g(m, n)$ which is defined as the longest derivation on the input $s^{n+1}(0) + (0 + (0 + \cdots + (0 + 0))\dots)$, where the number of zeros is $2m + 2$. Show that g grows faster than the Ackermann function.)

4. Finite bases

Murski's grupiod $\mathbb{M}$ is defined as a graph algebra of the graph



$M = \{a, b, \perp\}$, and the binary operation $\cdot$ is defined as

$$x \cdot y = \begin{cases} x & \text{if } (x, y) \in E \text{, or} \\ \perp & \text{otherwise.} \end{cases}$$

Problem 4.1. Prove that Murski's grupoid is not finitely based.

5. Abelian algebras

Problem 5.1 (easy)**.** Prove that every abelian algebra in congruence modular variety has a Mal'cev term. Use (without a proof) the characterization of congruence modular varieties by means of Gumm terms.

Problem 5.2. Let $\mathcal{V}$ be a variety, and let $\mathcal{V}_{\mathrm{ab}}$ denotes the class of all Abelian algebras in $\mathcal{V}$.

1. Prove that if $\mathcal{V}$ has a Mal'cev term then $\mathcal{V}_{\mathrm{ab}}$ is a subvariety of $\mathcal{V}$, and
2. find a variety $\mathcal{V}$ such that $\mathcal{V}_{\mathrm{ab}}$ is not a variety.