Preliminaries 3

Definition 1. Let $\mathbb{A} = (A; R_1, \dots, R_m)$ be a relational structure. The Constraint Satisfaction Problem over $\mathbb{A}$, or CSP($\mathbb{A}$) for short, is the following decision problem.

An input of CSP($\mathbb{A}$) (also called an *instance*) consists of a set V (of *variables*) and a set of constraints, where each constraint is a pair $(\mathbf{x}, R)$ with $\mathbf{x}$ a k-tuple of variables and R a k-ary relation from $\{R_1, \dots, R_m\}$.

The question is to decide whether the given instance has a *solution*, that is, a mapping $f : V \to A$ such that, for every contraint $(\mathbf{x}, R)$ in the instance, $f(\mathbf{x}) \in R$.

Less formally, an instance of CSP($\mathbb{A}$) is a formula of the form

$$\textit{variables} \in R_{i_1} \ \& \ \textit{variables} \in R_{i_2} \ \& \ \dots$$

and the question is whether we can assign elements of A to the variables so that the formula is true (" & " is understood as AND).

Example 2. Let $\mathbb{A} = (A; R_1, R_2, R_3)$, where R_1 is a ternary relation, R_2 is a unary relation and R_3 is a binary relation. An example of an instance of CSP($\mathbb{A}$) is

- $V = \{x_1, x_2, x_3, x_4, x_5\}$
- $((x_1, x_3, x_1), R_1), (x_2, R_2), ((x_5, x_1, x_2), R_1), ((x_3, x_1), R_3)$

A mapping $f : V \to A$ is a solution of this instance, if $(f(x_1), f(x_3), f(x_1)) \in R_1$ and $f(x_2) \in R_2$ and $(f(x_5), f(x_1), f(x_2)) \in R_1$ and $(f(x_3), f(x_1)) \in R_3$

In the less formal way we would write this instance as

$$(x_1, x_3, x_1) \in R_1 \ \& \ x_2 \in R_2 \ \& \ (x_5, x_1, x_2) \in R_1 \ \& \ (x_3, x_1) \in R_3$$

and solutions are evaluations of variables $x_1, \dots, x_5$ (in A) which make the formula true.