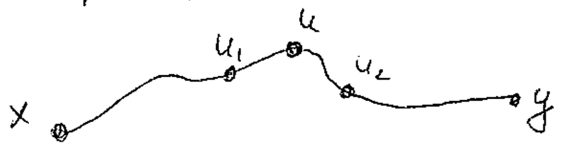
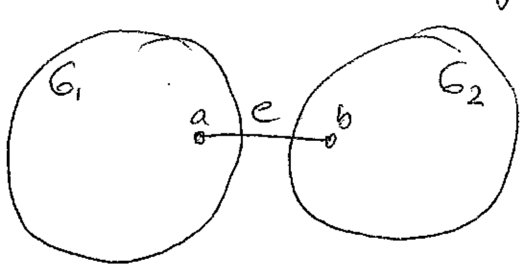
Recommended Problems 8 – Solutions

8.1 Let $u \in V(G')$ be arbitrary. We need to show that $G' - u$ is connected. So, let $x, y \in V(G' - u)$ be arbitrary vertices and we want to find a x,y-path in $G' - u$. Since G is connected, there exists a x,y-path P in G. If P doesn't contain u, then P is a x,y-path in $G' - u$ and we are done. Otherwise, let u_1 be the last vertex before u on the path P and let u_2 be the first vertex after u:



If $u_1u_2 \in E(G)$ then $x \sim u_1 - u_2 \sim y$ is an x,y-path in $G' - u$.
If $u_1u_2 \notin E(G)$ then $d_G(u_1, u_2) = 2$ therefor u_1, u_2 are adjacent in G' and, again, $x \sim u_1 - u_2 \sim y$ is an x,y-path in $G' - u$

8.2. Since $\kappa(G) = \kappa'(G)$ for 3-regular graphs, such a graph G has edge-connectivity 1, therefore it has a cut edge e. Let G_1, G_2 be the components of $G - e$



Each component has one vertex of degree 2 (a in G_1, b in G_2) and all the other vertices have degree 3. Then $|V(G_1)| \geq 4$ (and $|V(G_2)| \geq 4$). But $V(G_1)$ (or $V(G_2)$) cannot have 4 vertices, because the number of vertices of odd degree must be even. Therefore $|V(G_1)|, |V(G_2)| \geq 5$, thus $|V(G)| \geq 10$

On the other hand, there exists a 3-regular graph with 10 vertices and connectivity 1:

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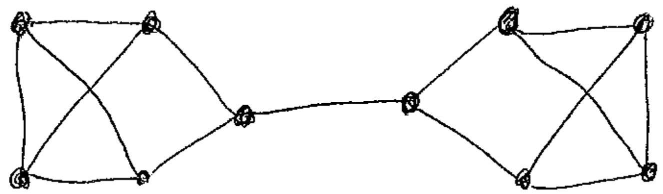
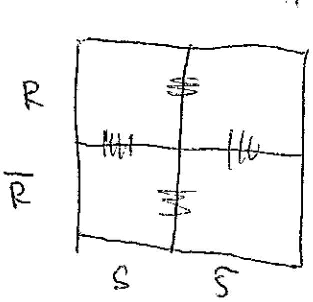
8.2 cntd



8.3 Let $[S,\bar S]$, $[R,\bar R]$ be different edge-cuts.



$[S,\bar S]$ contains all the edges between $S, \bar S$

$[R,\bar R]$ — " — $R, \bar R$

$[S,\bar S] \triangle [R,\bar R]$ contains all the edges between $S\cap R$, $\bar S \cap R$

and — " — $S\cap \bar R$, $\bar S\cap\bar R$

— " — $S\cap R$, $S\cap\bar R$.

$\bar S\cap R$, $\bar S\cap\bar R$

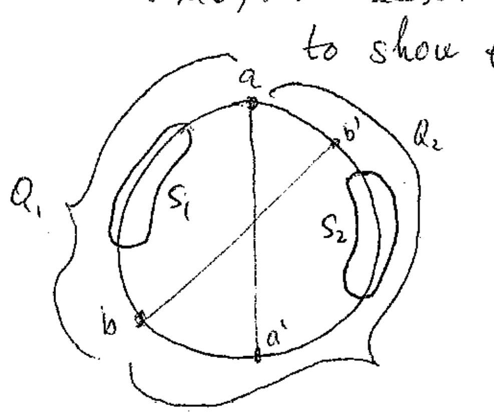
which is equal to the set of edges between $S\cap R \cup \bar S\cap\bar R$, $S\cap\bar R\cup\bar S\cap R$

$= [\, S\cap R\cup\bar S\cap\bar R,\ \overline{S\cap R\cup\bar S\cap\bar R}\,]$

this is an edge cut (both sets are nonempty since $[S,\bar S]$, $[R,\bar R]$ are different)

8.4 • $\kappa(G)\le k$ because if we delete all neighbors of a vertex, the graph becomes disconnected

• $\kappa(G)\ge k$ let $S\subseteq V(G)$, $|S|=k-1$ and let $a,b\in V(G)-S$. We need to show that a is connected to b in $G-S$.



Denote by Q_1 (Q_2, resp.) the vertices between a,b on the shorter (longer, resp.) part of the circle and let $S_1 = S\cap Q_1$, $S_2 = S\cap Q_2$.

If S_1 or S_2 doesn't contain $\frac{k-1}{2}$ consecutive vertices, then "we can jump across all the gaps" to get from a to b

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8.4 contd. Therefore we can assume that $|S_1| = |S_2| = \frac{k-1}{2}$ and both S_1 and S_2 are formed by consecutive vertices.

Let a' be the vertex opposite to a and b' the vertex opposite to b.

If S_2 contains no vertices between a and b' we can get from a to b' and then to b.

Similarly, if S_2 contains no vertices between b and a', a and b are connected.

Otherwise S_2 contains both a' and b' which is impossible as $d(a', b') = d(a, b) > \frac{k-1}{2}$ (otherwise a, b are adjacent).