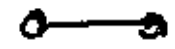
Recommended Problems 5 – Solutions

5.1 $\Rightarrow$ If T is a tree with degree sequence $(d_1, \dots, d_n)$, then $\sum d_i = 2|E(T)| = 2(n-1)$ (where the first equality is the Degree-Sum Formula and the second one uses the fact that a tree with n vertices has $n-1$ edges).

$\Leftarrow$ By induction on n

$n=2$ The only such sequence is $(1,1)$ which is the degree sequence of 

$n>2$ As $\sum d_i = 2n-2$ is a sum of n ~~positive integers~~ number, one of the summands must be smaller than 2. Therefore $d_i = 1$ for some i, WLOG $d_1 = 1$.

Also, one of the summands in $\sum d_i$ must be greater than 1 (otherwise $\sum d_i \le n < 2n-2$), WLOG $d_2 > 1$.

Consider a new sequence $(d_2 - 1, d_3, \dots, d_n)$.

Its elements are positive integers and their sum is

$$\underbrace{2n-2}_{\text{original sum}} - \underbrace{1}_{d_1 \text{ missing}} - \underbrace{1}_{d_2 \text{ decreased}} = 2(n-1) - 2.$$

By induction hypothesis there exists a tree T with $n-1$ vertices realizing $(d_2 - 1, d_3, \dots, d_n)$. We add a vertex and join this vertex to a vertex of T of degree $d_2 - 1$. The resulting graph is a tree (connected, correct number of edges) and realizes $(d_1, d_2, \dots, d_n)$.

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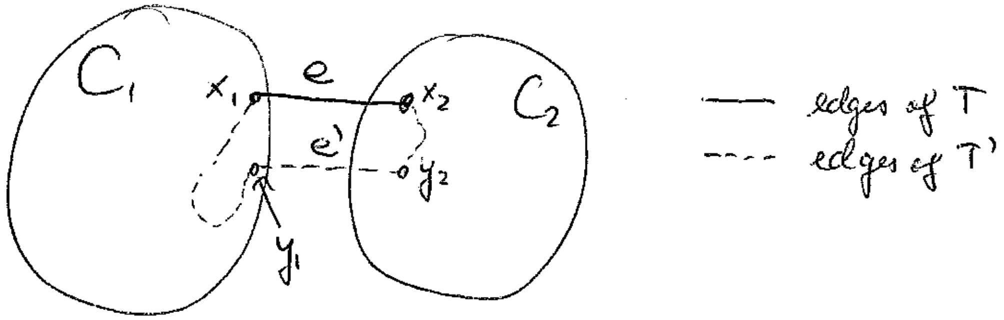
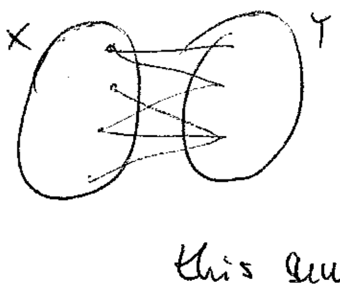
5.2 Let $e = x_1x_2 \in E(T) - E(T')$ and let C_i be the component of $T-e$ containing x_i, $i=1,2$. Consider the x_1x_2-path in T'. Take any edge $e' = y_1y_2$ in this path with $y_1 \in C_1$, $y_2 \in C_2$.



Clearly $e \neq e'$ (as $e \notin E(T')$) and $e' \notin E(T)$ (as C_1 and C_2 are components). Now $T-e+e'$ is connected (as e' joins the only two components of $T-e$) and has the correct number of edges, so it is a (spanning) tree.
The graph $T'+e-e'$ is acyclic (the only cycle in $T'+e$ is disconnected by removing e') and has the correct number of edges, so it is also a (spanning) tree.

5.3 Let X, Y be a bipartition of T with $|X| \geq |Y|$ (i.e. $|X| \geq \frac{n}{2}$ where n is the number of vertices of T).



The number of edges of T is the sum of the degrees of vertices in X. If all the vertices in X have degree at least 2 then this sum is at least $2 \cdot \frac{n}{2} = n$ which is impossible as every tree has $n-1$ edges. Therefore X contains a vertex of degree 1.

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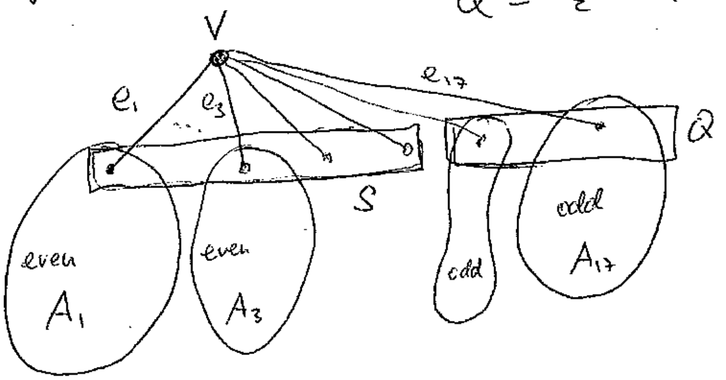
(5.4) By induction on $n = |V(T)|$. For $n=2$ it is clear, assume $n>2$.
Let v be a vertex which is not a leaf (it exists as $n>2$),
let $A_1, \dots, A_k$ be components of $T-v$ and let e_i be the edge
joining v to A_i. Let $S = \{ i : |V(A_i)| \text{ is even} \}$ and
$Q = \{ i : |V(A_i)| \text{ is odd} \}$



Since n is even, $|Q|$ is odd.

Observe that (*) No tree of odd order has spanning subgraph in which every vertex has odd degree.
(it's a corollary of the Degree-Sum Formula)

<u>Existence</u> ~~Take~~ Let H_i, $i \in S$ be a spanning subgraph of A_i
(A_i is a smaller even graph so we can use the induction hypothesis)
Let ~~Take~~ G_i, $i \in R$ be a spanning subgraph of $A_i + e_i$
($A_i + e$ is even, and smaller* because $d(v) > 1$ – that is why we took a non-leaf vertex v)

Take $G = \bigcup_{i \in S} H_i \cup \bigcup_{i \in R} G_i$. Each G_i contains the edge e_i
because otherwise $G_i - v$ would be a spanning subgraph of A_i
which violates (*). Therefore G is a spanning subgraph and every
vertex of G has odd order (for vertices other than v it follows from
the construction, and $d(v) = |Q|$ by the previous sentence)
odd!

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5.4 contd

<u>Uniqueness</u> Let G, G' by two spanning subgraphs with only odd-degree vertices.

No $e_i, i \in S$ can be an edge of G (G'), since otherwise an induced subgraph of G would be a spanning subgraph of $A_i + v$ ~~which is~~ (with all vertices of odd degree) which is impossible by (*)

~~On the other Then G and G' are~~

Then the subgraph of G (and G') induced by vertices of A_i is a spanning subgraph of A_i, therefore, by induction hypothesis, G and G' agree on A_i

On the other hand $e_i, i \in R$ is an edge of G (and G') for every $i \in Q$, since otherwise ~~the~~ an induced subgraph of G (G') would be a spanning subgraph of A_i (with all vertices of odd degree) which is again impossible by (*).

The subgraph of G (G') induced by vertices of $A_i + e_i$ is a spanning subgraph of $A_i + e_i$, therefore, by induction hypothesis, G and G' agree on A_i ($+e_i$)

We proved that G and G' agree on e_i's as well as A_i's, thus $G = G'$.