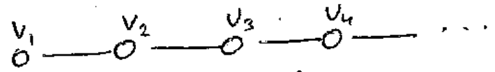
Recommended Problems 1 – Solutions

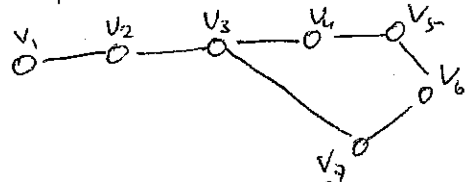
(1.1) We will study isomorphism classes of complements of such graphs. These are 7-vertex graphs in which every vertex has degree 2.

Such graphs are unions of disjoint cycles:

To see that consider an arbitrary vertex $\underline{v_1}$ and one of its neighbors $\underline{v_2}$. The vertex v_2 is adjacent to one more vertex $\underline{v_3}$, etc.

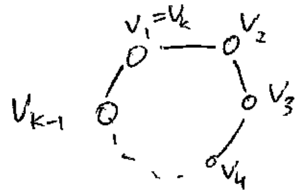


Take first v_k which is equal to some v_i, $i<k$.
Then $i=1$, otherwise v_i would have degree 3 (or more)



(picture for $k=7$, $i=3$) → impossible

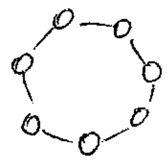
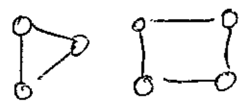
We have found a cycle $v_1, v_2, \ldots, v_{k-1}, v_1$



These vertices are adjacent to no other vertices (because of the degree)

It follows that the graph is a union of disjoint cycles.

There are 2 isomorphism classes of such graphs: C_7, C_3+C_4

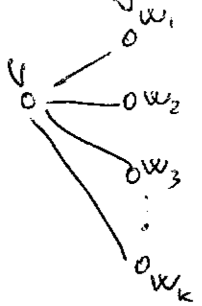
$\boxed{C_7}$  $\boxed{C_3+C_4}$ 

For the original question – there are 2 isomorphism classes (namely $\overline{C_7}$ and $\overline{C_3+C_4}$).

Recommended Problems 1 - Solutions

(1.2) The idea is similar ~~to~~ as in Homework Problem 1.3.

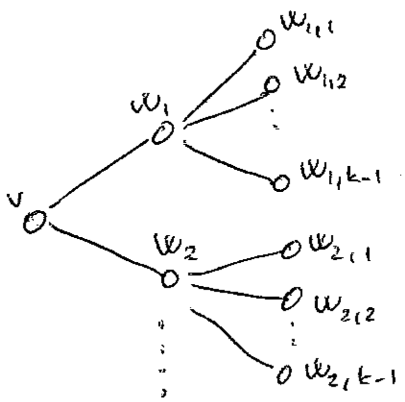
Take any $v \in V(G)$ and let $w_1, \ldots, w_k$ be its neighbors



each w_i is adjacent to $(k-1)$ vertices other than v, call them $w_{i,1}, w_{i,2}, \ldots, w_{i,k-1}$

now $w_{i,j} \neq w_k$ (for any i,j,k) otherwise we would get a cycle of length 3

and $w_{1,1}, w_{1,2}, \ldots, w_{1,k-1}, w_{2,1}, \ldots\ldots w_{k,k-1}$ are pairwise different, otherwise we would get a cycle of length 4

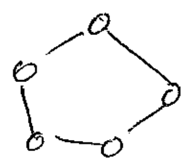


We have found k^2+1 different vertices:

$$\underbrace{v}_{1},\ \underbrace{w_1 \ldots w_k}_{k},\ \underbrace{\underbrace{w_{1,1} \ldots w_{1,k-1}}_{k-1}, \ \ldots\ \underbrace{w_{k,1} \ldots w_{k,k-1}}_{k-1}}_{k}$$

$$1 + k + (k-1)k = k^2+1$$

For $k=2$ an example of such graph is C_5



For $k=3$ we can take the Petersen graph

Recommended Problems 1 – Solutions

1.3

($\Rightarrow$) If a self-complementary graph with n vertices exists, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$

A self-complementary graph with n vertices has $\dfrac{\frac{n\cdot(n-1)}{2}}{2}$ edges (because it has the same number of edges as non-edges an the total number of unordered pairs of distinct vertices is $\frac{n(n-1)}{2}$)

Therefore $\frac{n\cdot(n-1)}{2}$ must be an even number

if $n \equiv 2 \bmod 4$, ie. $n = 4k+2$ for some k, then

$$\frac{n(n-1)}{2} = \frac{(4k+2)(4k+1)}{2} = (2k+1)(4k+1) \text{ which}$$

is odd

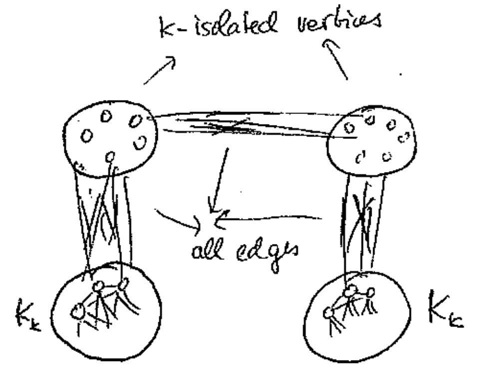
if $n \equiv 3 \bmod 4$, ie $n = 4k+3$ for some k, then

$$\frac{n(n-1)}{2} = \frac{(4k+3)(4k+2)}{2} = (4k+3)(2k+1) \text{ which is odd}$$

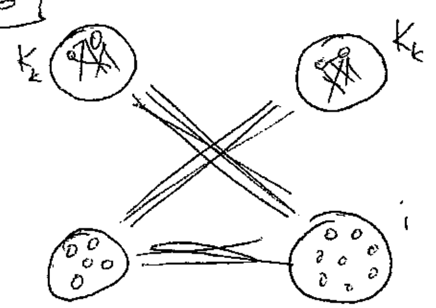
Hence $n \equiv 0$ or $1 \bmod 4$

($\Leftarrow$) For $n \equiv 0 \bmod 4$, ie. $n = 4k$ for some k we can take

$\boxed{G}$

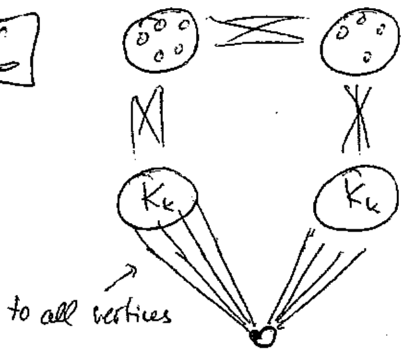


$\boxed{\overline{G}} \cong G$

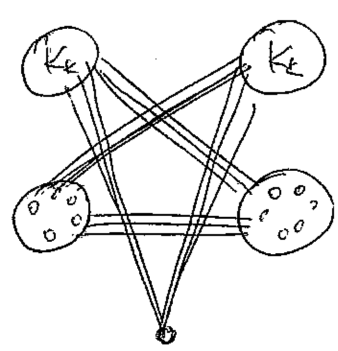


For $n \equiv 1 \bmod 4$, ie $n = 4k+1$ we add one vertex to G above

$\boxed{G}$



$\boxed{\overline{G}} \cong G$



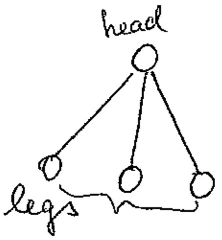
Recommended Problems 1 – Solutions

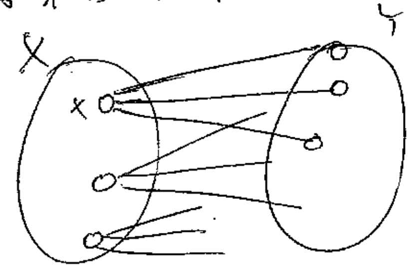
1.4

($\Leftarrow$) If G is bipartite, then G decomposes into claws

Let X, Y be the partite sets of G

For each vertex x in X we include one claw into decomposition so that x is its head



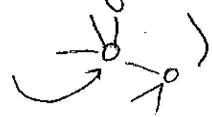
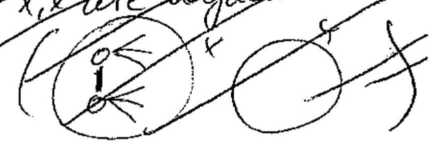


Because the degree of every vertex in X is 3, every edge incident to a vertex of X is covered.
But every edge is incident to a vertex of X and clearly no edge is covered twice, so this is a decomposition

($\Rightarrow$) If G decomposes into claws, then G is bipartite

(*) Let $X = \{u \in V(G);\ u \text{ is a head of some claw of the decomposition}\}$

$Y = \{u \in V(G);\ u \text{ is a leg of some claw of the decomposition}\}$

Now
- X, Y are disjoint as no vertex can be simultaneously head of one claw and leg of another claw (because of degrees )
- $X \cup Y = V(G)$ as every vertex is in the decomposition
- ~~X is independent. as If $x, x' \in V(G)$ and x, x' are adjacent then the degree of x is at least 4~~ 

better →
- ~~Y is independent.~~
- X and Y are independent: Every edge xy is in some claw, therefore one of the vertices is a head (and belongs to X) and the other one is a tail (and lies in Y).

(*) we take a decomposition of G

Therefore X, Y are partite sets.