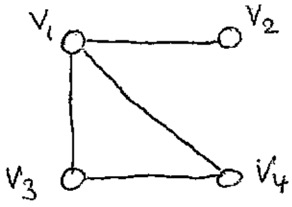
1) [12 marks] In this problem (and only this one) you do not need to justify your answer.

a) Find the adjacency matrix of the graph below.



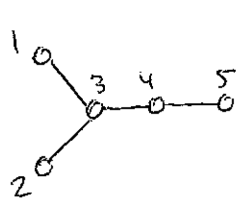
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

b) What is the number of simple graphs with vertex set $V = \{a, b, c, d, e\}$?

# of unordered pairs of distinct vertices is $\binom{5}{2} = 10$

# of simple graphs with $V(G) = \{a, b, c, d, e\}$ is $2^{10} = 1024$ //

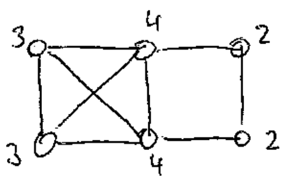
c) Find a simple graph containing a maximal path which does not have maximum length.



132 is a maximal path
it doesn't have maximum length
(the path 1345 is longer)

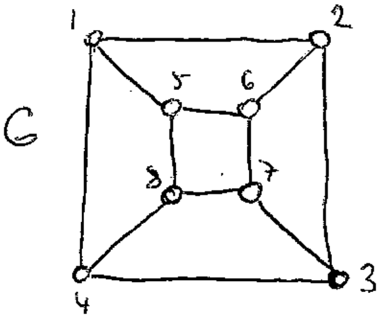
d) Is 2,2,3,4,3,4 a graphic sequence? If the answer is yes, draw a simple graph realizing this sequence.

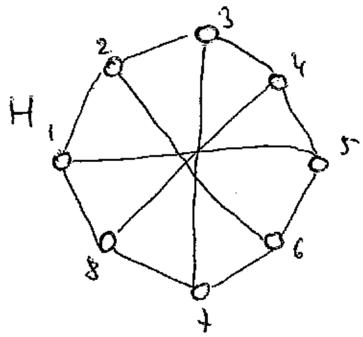
Yes



(Can be obtained by the method discussed in class, for instance)

2) [8 marks] a) Determine whether the graphs below are isomorphic.



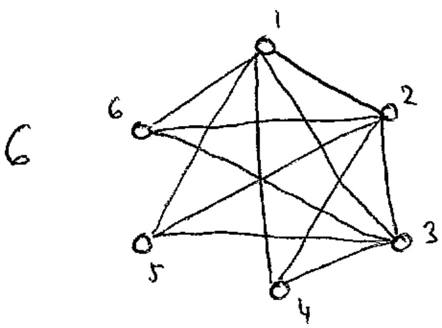


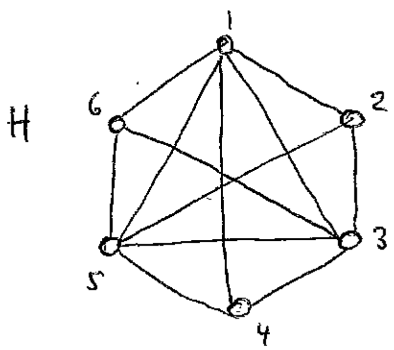
G is bipartite as $\{1,3,6,8\}$ and $\{2,4,5,7\}$ are disjoint independent sets with union $V(G)$

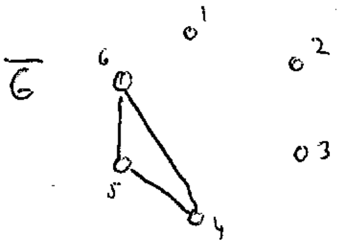
H is not bipartite, as it contains an odd cycle $(1,2,3,4,5)$

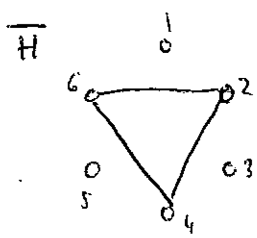
Therefore G is not isomorphic to H

b) Determine whether the graphs below are isomorphic.









$\overline{G} \simeq K_3 + 3K_1$

$\widetilde{H} \simeq K_3 + 3K_1$

Therefore $\overline{G} \simeq \widetilde{H}$, so $\boxed{G \simeq H}$

(An example of an isomorphism $G \to H$ is the bijection $f: V(G) \to V(H)$ defined by $f(1)=1$, $f(2)=3$, $f(3)=5$, $f(4)=4$, $f(5)=6$, $f(6)=2$)

3) [24 marks] In the following problems you may use facts proved in class. In this case state the theorem you use (but do not prove it).

a) A graph G has 11 edges and its degree sequence is $3, 1, 4, 4, 4, 4, x$. Find x.

Theorem: For every graph G, $\sum_{v \in V(G)} d(v) = 2|E(G)|$

$$3+1+4+4+4+4+x = 2 \cdot 11$$
$$x = 2$$

b) Does there exist a simple bipartite graph G of girth 7?

Theorem: A graph is bipartite if and only if it has no odd cycle

G has girth 7, so G contains a cycle of length 7, therefore G is not bipartite (using $\Rightarrow$ of the theorem)

c) Does every simple graph with 423 edges contain a bipartite subgraph with 175 edges?

Theorem: Every loopless graph G has a bipartite subgraph with at least $\frac{|E(G)|}{2}$ edges.

$\boxed{\text{YES}}$

By the theorem, a graph with 423 edges contains a bipartite graph with at least 212 edges. By deleting some of them (which does not destroy bipartiteness) we obtain a bipartite subgraph with 175 edges.

d) Determine the values of n such that K_n has a $\underbrace{\text{closed trail containing all edges}}$. $\to$ Eulerian $=$ Circuit

Theorem: Let G be a connected graph. Then G has an Eulerian circuit if and only if G is even.

K_n is connected. The degree of every vertex is $n-1$, therefore K_n is even iff n is odd.

By the theorem, K_n has an Eulerian circuit iff <u>n is odd</u>.

e) What is the minimum number of trails needed to decompose the Petersen graph?

Theorem: Let G be a connected nontrivial graph with exactly $2k$ odd vertices. The minimum number of trails that decompose G is $\max\{k,1\}$.

The Petersen graph is connected, nontrivial and has 10 vertices of odd degree (3).

By the theorem, the minimum number of trails that decompose G is $\max\{1,5\} = \underline{5}$.

f) Does there exist a simple graph with 20 vertices and 123 edges which does not contain K_3 as an induced subgraph?

Theorem: The maximum $|E(G)|$ in an n-vertex Δ-free simple graph G is $\lfloor \frac{n^2}{4} \rfloor$.

$\boxed{\text{NO}}$

For $n=20$ the theorem implies that a 20-vertex Δ-free graph has at most $\lfloor \frac{20^2}{4} \rfloor = 100$ edges.

Therefore a graph with 20 vertices and 123 edges contains an induced subgraph isomorphic to Δ.

4) [6 marks] Prove that the complement of a simple disconnected graph must be connected.

See the solutions to the recommended problems.