Homework 7 - Solutions

7.1 Let M_1, M_2 be perfect matchings of a tree G and consider their symmetric difference ~~the~~ graph H with vertex set $V(G)$ and edge set $M_1 \triangle M_2$.

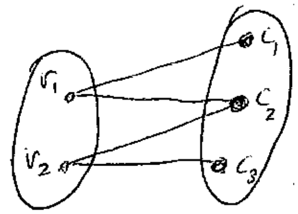
<u>$d_H(v) = 0$ or 2 for every $v \in V(H)$</u> : The vertex v is an endpoint of exactly one edge m_1 in M_1 (as M_1 is a matching and it is perfect). Similarly v is an endpoint of exactly one edge m_2 in M_2. If $m_1 = m_2$ then $d_H(v) = 0$ otherwise $d_H(v) = 2$

It follows that H is a ~~diff.~~ disjoint union of cycles and isolated vertices. But $E(H) \subseteq E(G)$ and G contains no cycle (it is a tree), therefore H contains no cycle and H contains only isolated vertices. This precisely means that $M_1 = M_2$.

7.2 Let $A = (a_{ij})_{i=1\ldots m,\, j=1\ldots n}$ be a $0,1$-matrix (with m rows and n colums). Consider the $\{r_1,\ldots,r_m\}, \{c_1,\ldots,c_n\}$-bigraph G where $r_i \leftrightarrow c_j$ iff $a_{ij} = 1$

<u>Example:</u>

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad G =$$



- A set ~~of~~ $a_{i_1 j_1}, a_{i_2 j_2}, \ldots a_{i_k j_k}$ of ones in A is independent iff the corresponding set $r_{i_1} c_{j_1}, \ldots r_{i_k} c_{j_k}$ of edges in G is a matching (because two entries of A lie in a common line if the corresponding edges share a vertex.). Therefore the maximum number of independent 1s in A is equal to the size of maximum matching in G

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- A set of rows $i_1, \dots i_k$ and columns $j_1, \dots j_\ell$ together contain all the 1s iff $\{r_{i_1}, \dots r_{i_k}, c_{j_1}, \dots c_{j_\ell}\}$ is a vertex cover of G (this is also immediate).

  Therefore the minimum number of lines that together contain all the 1s is equal to the minimum size of a vertex cover of G.

Now it is enough to apply the König–Egerváry theorem.

7.3 Consider the X,Y-bigraph G, where $X = \{1, 2, \dots, m\}$ and $i \leftrightarrow y$ iff $y \in A_i$. Selecting elements $a_1 \in A_1, a_2 \in A_2, \dots, a_m \in A_m$ can be viewed as selecting edges $1 \leftrightarrow a_1, 2 \leftrightarrow a_2, \dots, m \leftrightarrow a_m$ in G. The elements $a_1, \dots, a_m$ are distinct iff ~~the~~ these m-edges form a matching.

Therefore $\underline{A}$ has an SDR iff G has a matching which saturates X. By Hall's theorem, the latter happens iff $|N_G(S)| \geq |S|$ for every $S \subseteq \{1, 2, \dots, m\}$. But, by the definition of G,

$$N_G(S) = \bigcup_{i \in S} A_i .$$

<u>Example</u>

$Y = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, $\underline{A} = (A_1, A_2, A_3)$, $A_1 = \{1, 2\}$, $A_2 = \{2\}$, $A_3 = \{2, 3, 4\}$