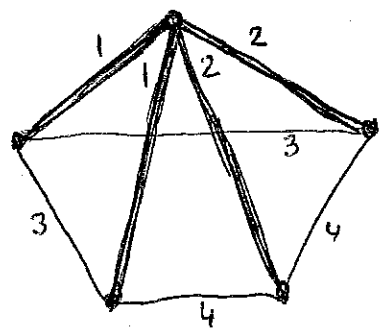
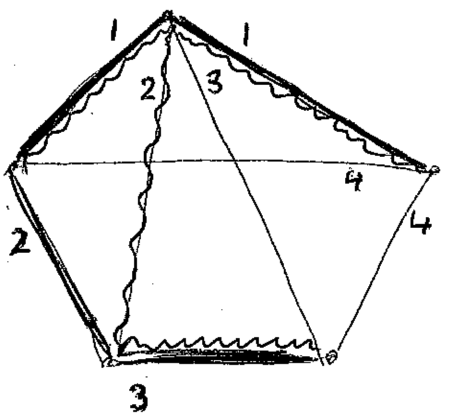
Homework 6 – Solutions

(6.1) unique:



the spanning tree using the four edges with smallest weight (1,1,2,2) is the unique minimum-weight spanning tree since every spanning tree has 4 edges and every other selection of edges has bigger sum of weights than $1+1+2+2$.

not unique:

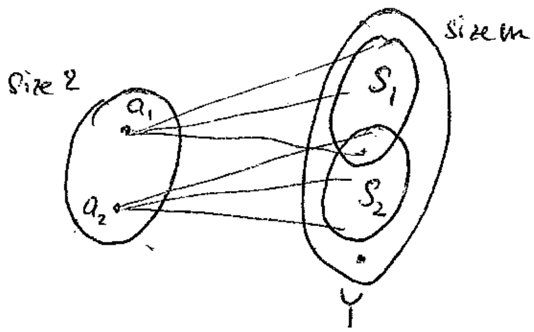


— 1st spanning tree

∼∼ 2nd one

both are minimum-weight spanning trees since they were obtained by Kruskal's algorithm

(6.2) Let a_1, a_2 denote the vertices in the two-element partite set. Consider an arbitrary spanning tree T of this graph and let S_i^T be the set of neighbors of a_i, $i=1,2$



- $S_1^T \cup S_2^T = Y$ (where Y denotes the partite set of size m) otherwise T is disconnected (the vertices in $Y-(S_1 \cup S_2)$ are not isolated in T)
- $S_1^T \cap S_2^T \neq \emptyset$ otherwise T is disconnected (a_1 would not be connected to a_2)
- $|S_1^T \cap S_2^T| = 1$. If $S_1^T \cap S_2^T \supseteq \{c, d\}$, $c \neq d$ then $a_1 - c - a_2 - d - a_1$ is a cycle in T.

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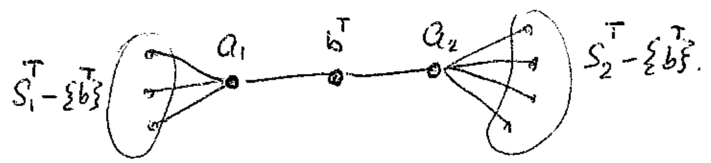
6.2 cntd. On the other hand, if $S_1, S_2 \subseteq Y$ are such that $S_1 \cup S_2 = Y$ and $|S_1 \cap S_2| = 1$ then the subgraph obtained by joining a_1 to all the vertices of S_1 and a_2 to all vertices of S_2 is a tree (it has the correct number of edges .. $m+1$, and it is connected).

We have found a bijection between the set of spanning trees and the set of pairs $S_1, S_2 \subseteq Y$ with $S_1 \cup S_2 = Y$, $|S_1 \cap S_2| = 1$. Each such a pair is determined by the element b in $S_1 \cap S_2$ and $S_1 - \{b\} \subseteq Y - \{b\}$ (as $S_2 = Y - (S_1 - \{b\}) \cup \{b\}$)
so the number of such pairs is

$$\underset{\text{selection of a}}{m} \cdot \underset{\text{selection of a subset of } Y - \{b\}}{2^{m-1}}$$

The number of spanning trees of $K_{2,m}$ is $m \cdot 2^{m-1}$.

In the previous part we assigned to every spanning tree T a pair of subsets $S_1^T, S_2^T \subseteq Y$ such that $S_1^T \cup S_2^T = Y$, $S_1^T \cap S_2^T = \{b^T\}$. The tree T can be drawn as follows:



Obviously if $|S_1^T - \{b^T\}| = |S_1^{T'} - \{b^{T'}\}|$ (and thus $|S_2^T - \{b^T\}| = |S_2^{T'} - \{b^{T'}\}|$)
then T and T' are isomorphic.
Also, if $|S_1^T - \{b^T\}| = |S_2^{T'} - \{b^{T'}\}|$, then T and T' are isomorphic
(by flipping the tree above)
Therefore the isomorphism class of T is completely determined by the size of smaller of the sets $S_1^T - \{b^T\}$ and $S_2^T - \{b^T\}$. These two disjoint sets have altogether $m-1$ elements, so the possible sizes of the smaller of them are $0, 1, 2, \dots, \lfloor \frac{m-1}{2} \rfloor$ ($\lfloor \frac{m-1}{2} \rfloor + 1$ possibilities)

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6.2 cont'd 2 The last thing we need to observe is that if T and T' have a different size of the smaller of the sets, then they are not isomorphic. Indeed, if the size is i then the degree sequence is

$$\underbrace{1, 1, \ldots, 1}_{m-1}, \underset{a_1/a_2}{1+i}, \underset{a_1/a_2}{m-i}, \underset{b}{2}$$

These sequences are essentially different for different i's. ("essentially" means that they don't just have different ordering).

The number of isomorphism classes of spanning trees of $K_{2,m}$ is $\left\lfloor \frac{m-1}{2} \right\rfloor + 1 = \left\lceil \frac{m}{2} \right\rceil$.

6.3 A sequence $(a_1, \ldots, a_{n-2})$, $a_i \in \{1, 2, \ldots, n\}$ is the Prüfer code of such a tree iff all but exactly 3 of the numbers $\{1, 2, \ldots, n\}$ appear in the sequence (by the lemma about degrees and Prüfer code from class). Therefore one number repeats twice in the sequence and there are no other repetitions.

The number of such sequences is:

$$\binom{n}{3} \cdot (n-3) \cdot \frac{(n-2)!}{2}$$

$\binom{n}{3}$: # of ways how we can choose $\{a_1, \ldots, a_{n-2}\}$

$(n-3)$: # of choices for the repeated vertex

$(n-2)!$: number of permutations of $a_1, \ldots, a_{n-2}$

2: counting each sequence 2

This can be simplified to $\frac{n!(n-3)(n-2)}{12}$.

The number of such trees is $\frac{n!(n-3)(n-2)}{12}$.