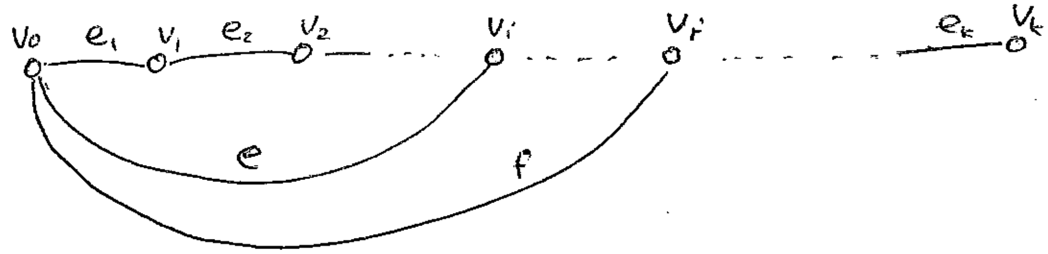
Homework 3 - Solutions

(3.1) Let $P = v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ be a maximal path in G. Since v_0 has degree ~~3/ there~~ at least 3 there are 2 (different) edges e, f incident to v_0, other than e_1. ~~Let~~ As P is maximal, both endpoints of e (and f) lie on the path P. Let v_i (v_j resp.) denote the endpoint of e (f, resp.) different from v_0. (Recall that G is loopless, so v_i, v_j is well defined and $i, j > 0$.)



~~If~~ Let's assume that $i \le j$ (otherwise we swap e and f)

If i is ~~even~~ odd, then $v_0, e_1, v_1, e_2, \ldots v_i, e, v_0$ is a cycle of even length (namely $i+1$)

If j is odd, then $v_0, e_1, v_1, \ldots, v_j, \cancel{e} f, v_0$ is a cycle of even length (namely $j+1$)

If both i and j are even, the $v_0, e, v_i, e_{i+1}, v_{i+1}, \ldots, v_j, f, v_0$ is a cycle of even length (namely $j-i+2$)

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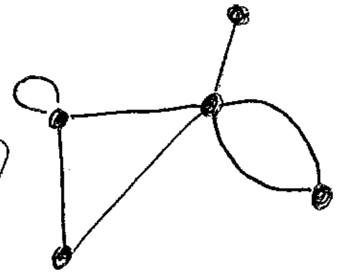
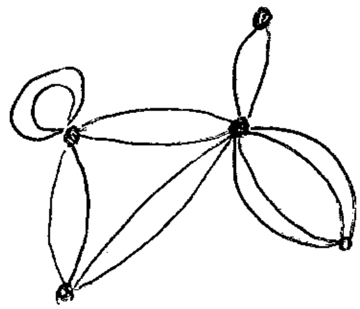
(3.2) Let G be connected, nontrivial graph. We consider the graph H obtained from G by replacing each edge e by two edges $e^{(1)}, e^{(2)}$ with the same endpoints as e.

G



H



The graph H is even (the degree of a vertex v in H is twice the degree of v in G) and connected.
Therefore, by the theorem characterizing Eulerian graphs, H has a closed trail

$$T = v_0, e_1^{(i_1)}, v_1, e_2^{(i_2)}, v_2, \dots\dots, v_0$$

which contain every edge of H exactly once.
Now

$$T' = v_0, e_1, v_1, e_2, v_2, \dots, v_0$$

is a closed trail containing every edge of G exactly twice (one appearence of an edge e corresponds to $e^{(1)}$ in T, the other one to $e^{(2)}$ in T)

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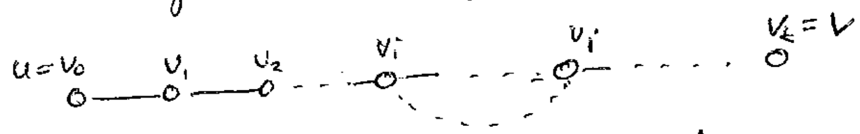
(3.3) a) true

The claim is clearly true if $|V(G)| \leq 2$, so let's assume $|V(G)| \geq 3$

We will use the following claim

Claim For any $u, v \in V(G)$, any u,v-path of minimum length is an induced subgraph

Proof: Let $u = v_0, v_1, \ldots, v_k = v$ be vertices on a u,v-path of minimum length (consecutively):



If G contains an edge $v_i \leftrightarrow v_j$ $(i < j-1)$, then $v_0, v_1, \ldots, v_i, v_j, \ldots, v_k$ is a shorter path, thus there is no such edge and the path is an induced subgraph □

Now, let $u \in V(G)$ be arbitrary

- ~~As~~ Case 1: There exists $v \in V(G)$, $u \neq v$ such that $uv \notin E(G)$. G is connected, so there ~~is~~ exists a u,v-path. Consider a u,v-path P of minimum length. The vertices u,v are not adjacent, hence P has length at least 2. Then the first three vertices of P induce a subgraph isomorphic to P_3 which contains u.

- Case 2: u is adjacent to every other vertex of G. Since G is not complete, there exist v_1, v_2 such that $v_1v_2 \notin E(G)$, $v_1 \neq v_2$. By the assumption, $|\{u, v_1, v_2\}|$ ~~are~~ $= 3$; $u \leftrightarrow v_1$, $u \leftrightarrow v_2$, therefore the subgraph induced by $\{u, v_1, v_2\}$ is isomorphic to P_3.

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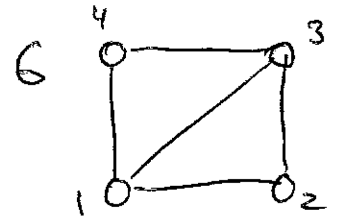
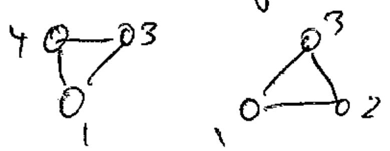
3.3 b) false



the graph above is connected, simple and not complete.
But the edge $1 \leftrightarrow 3$ doesn't belong to an induced subgraph isomorphic to P_3
(the only induced subgraphs containing the vertices 1 and 3 are the triangles



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