Homework 2 - Solutions

2.1 Vertices 1, 11, 13 are isolated

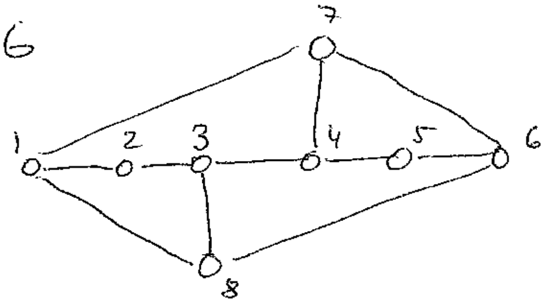
The remaining (12) vertices lie on the path

P = 7, 14, 2, 4, 6, 8, 10, 5, 15, 3, 9, 12

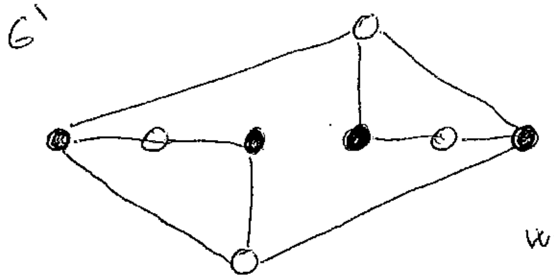
Therefore

- they all are in one component, so the number of components is 4
- the maximum length of a path in G is 11: there cannot be any longer path as all the vertices of a path must be in the same component and P contains all vertices of the largest component ~~(so there can~~

2.2



- G has 11 edges, it is not bipartite, because 1,2,3,4,7,1 is a cycle of odd length
- G has a bipartite subgraph G' with 10 edges; the bipartition is indicated on the picture (solid vertices form one partite set, the other vertices form the other partite set)



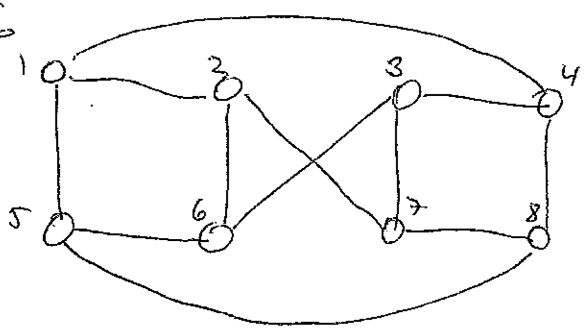
- G has no other bipartite subgraph with 10 edges: the cycles 1,2,3,4,7,1 and 3,4,5,6,8,3 have both odd length, so we have to remove at least one edge from both of them to make G bipartite. But the only edge they share is 3↔4, thus removing 3↔4 is the only way how to make G bipartite with only one edge deletion.

→ G' is the unique bipartite subgraph of G with the maximum number of edges.

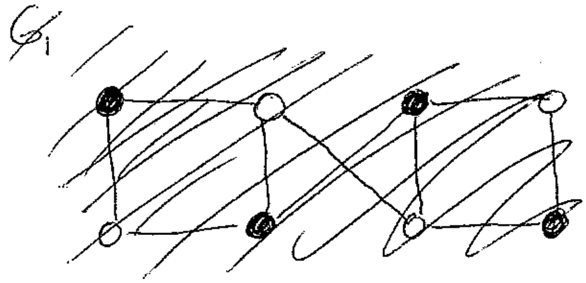
Homework 2 - Solutions

2.2 contd.

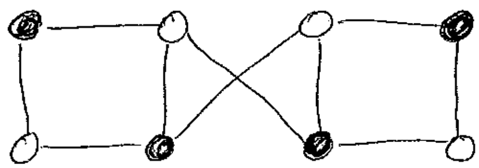
G



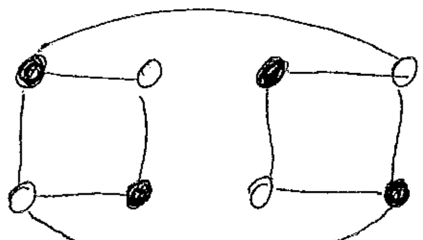
- G has 12 edges
- G has (at least) 2 different ~~sub~~ bipartite subgraphs with 10 edges (the bipartition is indicated on the picture, as before)



G_1



G_2



- G has no bipartite subgraphs with 11 or 12 edges:

  the cycles 5,6,3,7,8,5 and 1,2,6,3,4,1 have odd length, so we have to delete at least one edge from both of them to make G bipartite. The only edge they share is $3 \leftrightarrow 6$, so if we want to delete only one edge, we have to delete this one. But it is not enough - the resulting subgraph is still not bipartite (for instance, 1,2,7,3,4,1 is an odd cycle) It follows that we have to delete at least 2 edges to make G bipartite.

$\rightarrow$ G_1 is a bipartite subgraph of G with maximum number of edges. It is not unique.

Homework 2 - Solutions

(2.3)

(=>) Let G be a bipartite graph, let X, Y be its partite sets. Let H be an arbitrary subgraph of G. We have to show that H has an independent set ~~wit~~ of size at least $\frac{|V(H)|}{2}$.
This is true, as both $X \cap V(H)$ and $Y \cap V(H)$ are independent sets, and, as $(X \cap V(H)) \cup (Y \cap V(H)) = V(H)$, at least one of them has size $\geq \frac{|V(H)|}{2}$.

(<=) Let G be a graph such that every subgraph H of G has an independent set ~~with~~ of size at least $\frac{|V(H)|}{2}$. We have to show that G is bipartite. ~~We sh~~ It is enough to show that G contains no odd cycle (we are using the characterization of bipartite graphs from class).
Assume the converse, ie. G contains an odd cycle H.
H is a subgraph of G, so it must have an independent set of size at least $\frac{|V(H)|}{2}$. But this is absurd as every subset of $V(H)$ with at least $\frac{|V(H)|}{2}$ vertices clearly contains two adjacent vertices.

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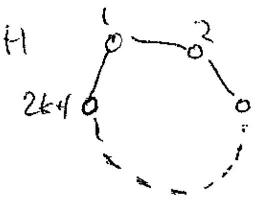
this "clearly" deserves explanation. Denote the vertices of H by $1, 2, \dots, 2k+1$ and let X be a set of size ~~at least~~ $k+1$, say

H 

$X = \{i_1, i_2, \dots i_{k+1}\}$ $i_1 < i_2 < \dots i_{k+1}$.

If X is independent, then $i_2 \geq i_1 + 2$, $i_3 \geq i_2 + 2 \geq i_1 + 4$ $\dots$ $i_{k+1} \geq i_k + 2 \geq \dots \geq i_1 + 2k$

But $i_{k+1} \leq 2k+1$, hence $i_1 = 1$ and $i_{k+1} = 2k+1$, and then X is not independent.

If X has even more than $k+1$ elements we can apply the argument above to a $(k+1)$-element subset of X.