Homework 1 - Solutions

(1.1) We prove the contrapositive, i.e. "If a graph can be decomposed into three paths, then it has at most six vertices of odd degree."

So, let G be a graph which can be decomposed into three paths P_1, P_2, P_3.

Internal vertices of a path have degree 2. Therefore, if a vertex $\underline{v} \in V(G)$ is an endpoint of no of the paths P_1, P_2, P_3 then $\underline{v}$ has even degree (the degree is twice the number of paths it is contained in).

Put differently, if v has odd degree, then it must be an endpoint of at least one of the paths P_1, P_2, P_3. These paths have altogether 6 endpoints, thus there are at most 6 vertices of odd degree.

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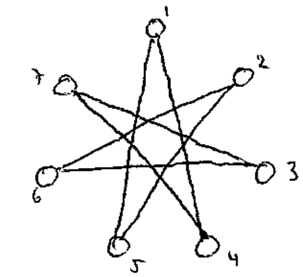
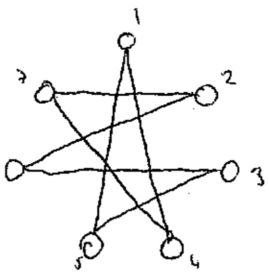
(1.2) The graph G_4 has a vertex (the middle one) of degree 6 while in all the other graphs all vertices have degree 4.
It follows that G_4 is not isomorphic to any of the other graphs.

We will consider the complements of the remaining graphs
(recall that $A \cong B$ iff $\bar{A} \cong \bar{B}$)

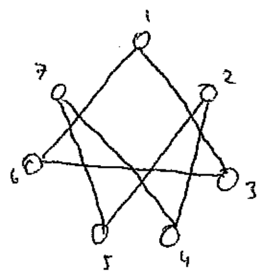
$\boxed{\overline{G_1}}$



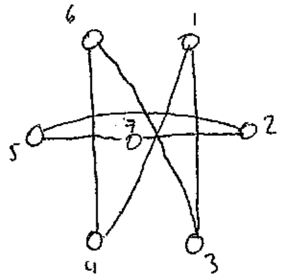
$\boxed{\overline{G_2}}$



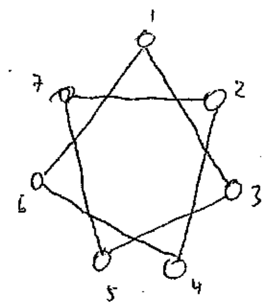
$\boxed{\overline{G_3}}$



$\boxed{\overline{G_5}}$



$\boxed{\overline{G_6}}$

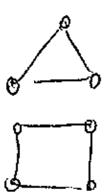


Now we see that $\overline{G_1}$ is isomorphic to C_7 (to see this order the vertices along a circle in the order $1,4,7,3,6,2,5$)

also $\overline{G_2} \simeq C_7$ $(1,4,7,2,6,3,5)$
and $\overline{G_6} \simeq C_7$ $(1,3,5,7,2,4,6)$
Therefore $\overline{G_1} \simeq \overline{G_2} \simeq \overline{G_6}$ $(\simeq C_7)$ and hence $G_1 \simeq G_2 \simeq G_6$ $(\simeq \overline{C_7})$

Graphs $\overline{G_3}$ and $\overline{G_5}$ are both isomorphic to $C_3 + C_4$, so $G_3 \simeq G_5$ $(\simeq \overline{C_3+C_4})$
and neither of them is isomorphic to ~~the remaining graphs~~ G_1 (or G_2, G_6)
(as $\overline{G_3}$ is ~~connected~~ disconnected and $\overline{G_1}$ is connected)

$\boxed{C_3+C_4}$



Isomorphism classes:

G_5 $\quad$ G_1 G_2 G_6 $\quad$ G_3 G_5

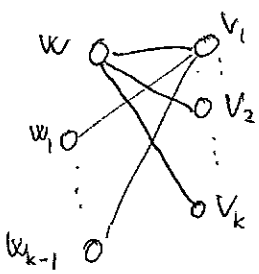
Homework 1 – Solutions

(1.3) Let G be a graph with girth 4 in which every vertex has degree k.

We show that $|V(G)| \geq 2k$.

Take an arbitrary $w \in V(G)$. As the degree of w is k, w has k neighbors, call them $v_1, v_2, \dots, v_k$.

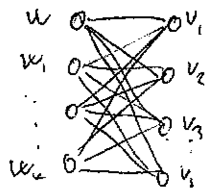
Now let's look at the vertex v_1. This vertex can't be adjacent to any of the vertices $v_2, \dots, v_k$, otherwise the girth would be less than 4 (If, say, $v_1 v_i \in E(G)$, then w, v_1, v_i form a triangle)



But v_1 is adjacent to $(k-1)$ vertices other than w, call them $w_1, \dots, w_{k-1}$ (and these vertices are different from $v_2, \dots, v_k$ as argued above).

We have found $2k$ pairwise different vertices, so $|V(G)| \geq 2k$ (namely $\underbrace{w}_{1}, \underbrace{v_1, \dots, v_k}_{k}, \underbrace{w_1, \dots, w_{k-1}}_{k-1}$)

Now suppose $|V(G)| = 2k$, thus there are no vertices other than $w, w_1, \dots, w_k, v_1, \dots, v_k$. The vertex v_2 has degree k and it is not adjacent to any of the vertices $v_1, v_3, v_4, \dots, v_k$ (the same argument as above – we would get a triangle), so it must be adjacent to all of the vertices $w_1, \dots, w_k$. Similarly, $v_3, \dots, v_k$ are adjacent to all of the vertices $w, w_1, \dots, w_{k-1}$. We have proved that G contains the graph



$(\cong K_{k,k})$ as a subgraph. But in $K_{k,k}$ all the vertices have degree k, therefore G doesn't contain any other edge.

The only such graph with $2k$ vertices is $K_{k,k}$ (up to isomorphism).