Universal Algebra 1 – Exercises 9

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Exercise 1. Let $\mathbf{G}$ be a group and let $\mathcal{A}$ denote the variety of Abelian groups. Denote by e the identity element and by $[x,y] = xyx^{-1}y^{-1}$ the left commutator.

(i) Prove that
$$e/\lambda_{\mathcal{A}}^{\mathbf{G}} = \mathrm{Sg}^{\mathbf{G}}\left(\{[x,y] \colon x, y \in G\}\right).$$

(ii) Prove that the variety $\mathcal{A} \cdot \mathcal{A}$ is defined by the group laws together with the identity
$$[[x,y],[z,w]] \approx e.$$

Exercise 2. Let $\mathcal{A}_n$ denote the variety of Abelian groups satisfying $x^n \approx e$.

(i) Prove that the variety $\mathcal{A}_3 \cdot \mathcal{A}_2$ is defined by the group laws together with the identities
$$x^6 \approx e, \qquad [x^2, y^2] \approx e, \qquad [x,y]^3 \approx e.$$

(ii) Prove that the variety $\mathcal{A}_2 \cdot \mathcal{A}_2$ is defined by the group laws together with the identity $(x^2y^2)^2 \approx e$.

Exercise 3. Let $\mathcal{Cr}_n$ denote the variety of commutative rings satisfying $x^n \approx x$, and let $\mathbb{F}_9$ denote a commutative ring which is a finite field of order 9. Prove that $\mathsf{V}(\mathbb{F}_9)$ is the variety defined by the axioms of $\mathcal{Cr}_9$ together with the identity $3x \approx 0$.