Universal Algebra 1 – Exercises 7

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Exercise 1. Let $\mathbf{A} = \langle A, \cdot \rangle$ be a binar. $\mathbf{A}$ is called *left-zero semigroup* if it satisfies $x \cdot y \approx x$. Similarly, $\mathbf{A}$ is called *right-zero semigroup* if it satisfies $x \cdot y \approx y$. Finally, $\mathbf{A}$ is called *rectangular band* if it satisfies the identities

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z), \qquad x \cdot x \approx x, \qquad (x \cdot y) \cdot z \approx x \cdot z.$$

(i) Let $\mathbf{A}$ be a rectangular band. Define

$$\lambda = \left\{ (x, y) \in A^2 \colon \forall z \in A,\ x \cdot z = y \cdot z \right\}.$$

Prove that $\lambda \in \mathrm{Con}(\mathbf{A})$ and that $\mathbf{A}/\lambda$ is a left-zero semigroup.

(ii) Prove that $\mathbf{A}$ is a rectangular band if and only if $\mathbf{A} \cong \mathbf{L} \times \mathbf{R}$ for some left-zero semigroup $\mathbf{L}$, and right-zero semigroup $\mathbf{R}$.

Exercise 2. Let $\mathbf{S} = \langle S, \cdot \rangle$ be a semilattice. A nonempty set I of S is called *ideal* if for every $s \in S$ and $a \in I$, both $a \cdot s$ and $s \cdot a$ are elements of I.

(i) Prove that, for every ideal I, the binary relation $I^2 \cup 0_S$ is a congruence on $\mathbf{S}$ (called the *Rees congruence* induced by I).

(ii) Let $a \in S$. Prove that $aS = \{a \cdot s \colon s \in S\}$ is an ideal of $\mathbf{S}$. Describe aS in terms of the ordering of $\mathbf{S}$.

(iii) Show that the only subdirectly irreducible semilattice is the two-element chain.

Exercise 3.* Prove that every chain is a directly indecomposable lattice.

Exercise 4.* Let $n \geq 1, k \geq 0$ and p be prime. Define the monounary algebras

$$\mathbf{C}_n = \langle \mathbb{Z}_n, (0\,1\,2\,\ldots\,n-1) \rangle$$
$$\mathbf{C}_n + 1 = \langle \mathbb{Z}_n, (0\,1\,2\,\ldots\,n-2)(n-1)) \rangle$$
$$\mathbf{P}_n = \langle \mathbb{Z}_n, u(x) = \min\{x+1, n-1\} \rangle$$

(i) Draw the graph of $\mathbf{C}_n, \mathbf{C}_n + 1, \mathbf{P}_n$.

(ii) Prove that $\mathbf{C}_{p^k}, \mathbf{C}_{p^k} + 1, \mathbf{P}_n$ are subdirectly irreducible.

(iii) Prove that $\mathbf{C}_{p^k}, \mathbf{C}_{p^k} + 1, \mathbf{P}_n$ are the only subdirectly irreducible monounary algebras.

(iv) Determine the minimal varieties in the lattice of subvarieties of the variety of monounary algebras.