Universal Algebra 1 – Exercises 6

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Exercise 1. Let $n \geq 1$ be an integer. Define $\mathbf{C}_n = \langle \mathbb{Z}_n, f_n \rangle$ to be the monounary algebra with the operation $f_n(x) = x + 1 \pmod{n}$.

(i) Describe $\mathbf{C}_n$ as a planar graph.

(ii) Draw the graph of $\mathbf{C}_n$ for $n \in \{2, \dots, 6\}$.

(iii) For which $n \in \{2, \dots, 6\}$ is $\mathbf{C}_n$ a simple algebra?

(iv) For which $n \in \{2, \dots, 6\}$ is $\mathbf{C}_n$ subdirectly irreducible?

(v) For which $n \in \{2, \dots, 6\}$ is $\mathbf{C}_n$ directly indecomposable?

Exercise 2. Let $\mathbf{A} = \langle \{0, 1, 2, 3, 4\}, f \rangle$ be the monounary algebra where the unary operation is described by $f \colon \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 2 \end{smallmatrix}\right)$

(i) Represent $\mathbf{A}$ as a graph.

(ii) Draw the lattice $\mathrm{Con}(\mathbf{A})$.

(iii) Is $\mathbf{A}$ subdirectly irreducible?

(iv) Is $\mathbf{A}$ directly indecomposable?

Exercise 3. Let $\mathbf{A} = \langle \{0, 1\}, +, f \rangle$ and $\mathbf{B} = \langle \{0, 1\}, +, g \rangle$ be the algebras of type $(2, 1)$, where $+$ denotes the addition modulo 2, and f, g are the unary operations $f(x) = x$ and $g(x) = x + 1$.

(i) Let $d \colon \mathbf{B}^2 \to \mathbf{A}$ be the map defined by $d(x, y) = x + y$. Show that d is a surjective homomorphism.

(ii) Let $\delta = \ker(d)$, and let η_i be the projection kernels of $\mathbf{B}$, for $i = 1, 2$. Show that $\{\eta_1, \eta_2\}$, $\{\eta_1, \delta\}$, and $\{\eta_2, \delta\}$ each form a pair of complementary factor congruences on $\mathbf{B}^2$.

(iii) Prove that $\mathbf{B} \times \mathbf{B} \cong \mathbf{B} \times \mathbf{A} \ncong \mathbf{A} \times \mathbf{A}$.

(iv) Deduce that the direct decomposition may be not unique.

Exercise 4. Find a pair of similar algebras, neither of which can be embedded into their product.

Exercise 5. Is the three-element chain subdirectly irreducible? If not, represent it as a subdirect product of subdirectly irreducible lattices.