Universal Algebra 1 – Exercises 3

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Exercise 1. Prove that every complete lattice is bounded.

Exercise 2. Find examples of

(i) A complete lattice with a complete sublattice.
(ii) A complete lattice with a sublattice which not a complete lattice.
(iii) A non complete lattice with a sublattice which is a complete lattice.
(iv) A complete lattice with a sublattice which is a complete lattice but not a complete sublattice.

Exercise 3. Let $\mathbf{L}$ be a complete lattice and let $a, b \in L$ be compact elements.

(i) Is $a \vee b$ compact?
(ii) Is $a \wedge b$ compact?

Exercise 4. Let C be a closure operator on a set A. Prove that $\mathbf{L}_C$ is closed under finite unions if and only if $C(X \cup Y) = C(X) \cup C(Y)$ for every $X, Y \in \mathcal{P}(A)$.

Exercise 5.* Let $\mathbf{M}$ be an algebraic complete lattice. Prove that there is an algebraic closure operator C on a set A such that $\mathbf{M} \cong \mathbf{L}_C$.

Exercise 6. Let X be a set, and let ϕ be the binary relation on $\mathcal{P}(X)$ defined by

$$(U, V) \in \phi \quad \overset{\text{def.}}{\iff} \quad U \cap V \neq \emptyset.$$

(i) Let $X = \{1, 2, 3, 4\}$. Compute $\mathcal{A}^{\blacktriangleright\blacktriangleleft}$ and $\mathcal{A}^{\blacktriangleleft\blacktriangleright}$ for $\mathcal{A} = \big\{\{1, 2\}, \{2, 3\}\big\}$ and $\mathcal{A} = \big\{\{1, 2\}, \{3\}\big\}$. Compare the results.
(ii) Prove that if the Galois correspondence is defined by a symmetric relation on a set, then the closure operators induced by it coincide.
(iii) [†]Prove that for every $\mathcal{A} \subseteq \mathcal{P}(X)$ we have $\mathcal{A}^{\blacktriangleright\blacktriangleleft} = \mathcal{P}_{\mathcal{A}}(X)$, where

$$\mathcal{P}_{\mathcal{A}}(X) := \{U \in \mathcal{P}(X) \colon \exists V \in \mathcal{A} \quad V \subseteq U\}$$

is the family of subsets of X that contain an element of $\mathcal{A}$.

[†]It may require the Axiom of Choice.