Universal Algebra 1 – Exercises 12

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22 December 2022, Prague

Exercise 1. Let $\mathcal{V}$ be a variety with a majority term m. Prove that $\mathcal{V}$ is congruence distributive (provide direct proof, without using Jonsson terms).

Exercise 2. An algebra $\mathbf{A}$ is called *arithmetical* if it is both congruence permutable and congruence distributive. Let $\mathbf{A}$ be an algebra. Prove that $\mathbf{A}$ is arithmetical if and only if for every $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ we have

$$\gamma \wedge (\alpha \circ \beta) \subseteq (\gamma \wedge \beta) \circ (\gamma \wedge \alpha).$$

Theorem (Jonsson)**.** *Let $\mathcal{V}$ be a variety of algebras. The following are equivalent.*

(a) $\mathcal{V}$ is congruence distributive.

(b) $\mathbf{F}_{\mathcal{V}}(3)$ is congruence distributive.

(c) There is a positive integer n and ternary terms $p_0, p_1, \ldots, p_n$ such that $\mathcal{V}$ satisfies the identities

(i) $p_i(x, y, x) \approx x$, for $0 \leq i \leq n$.

(ii) $p_0(x, y, z) \approx x$.

(iii) $p_n(x, y, z) \approx z$.

(iv) $p_i(x, x, y) \approx p_{i+1}(x, x, y)$, for i even.

(v) $p_i(x, y, y) \approx p_{i+1}(x, y, y)$, for i odd.

Exercise 3. In this step-by-step exercise we prove Jonsson's theorem. Let $\mathbf{F}$ denote the free algebra $\mathbf{F}_{\mathcal{V}}(3) = \mathbf{F}_{\mathcal{V}}(\{x, y, z\})$.

Step 1. Prove the trivial implication $(a) \implies (b)$.

Step 2. Consider the implication $(b) \implies (c)$. Define $\alpha = \mathrm{Cg}^{\mathbf{F}}(x, y), \beta = \mathrm{Cg}^{\mathbf{F}}(y, z), \gamma = \mathrm{Cg}^{\mathbf{F}}(x, z)$ and show that $(x, z) \in (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$.

Step 3. Use a characterization of the join of congruences and the generators of the free algebras to obtain a sequence of terms $u_i = p_i^{\mathbf{F}}(x, y, z)$.

Step 4. Use the freeness of $\mathbf{F}$ to prove that those terms satisfy the conditions in (c).

Step 5. Consider the implication (c) $\implies$ (a), and let $(a, b) \in (\alpha \vee \beta) \wedge \gamma$. Again, using the characterization of the join of congruences, find a chain of elements $c_0, \ldots, c_n$ such that $a = c_0 \, \alpha \, c_1 \, \beta \, c_2 \ldots$.

Step 6. Prove that $(p_i(a, c_j, b), p_i(a, c_{j+1}, b)) \in (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$ for $i \leq n$ and $j < k$.

Step 7. Prove that $(p_i(a, a, b), p_i(a, b, b)) \in (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$ for $i < n$.

Step 8. Prove that $(p_i(a, b, b), p_{i+1}(a, b, b)) \in (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$ for $i < n$ and conclude.