(UA) [12.1]

(WAS)

(I) LATTICES

complete lattices, closure operators, Galois correspondences

(II) SEMANTICS

- H, S, P, varieties
- direct & subdirect decompositions

(III) SYNTAX

- terms, free algebras, identities
- Mod-Id Galois correspondence

(IV) CLONES & COCLONES

- clones, free algebras as clones of term operations
- coclones, Pol-Inv Galois correspondence

NOW

(V) MAL'TSEV CONDITIONS

- Mal'tsev operation, congruence permutability, rectangularity
- majority, 2-decomposability

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Mal'tsev condition

(more precisely, this is called strong Mal'tsev condition)

= condition for a variety $\mathcal{V}$ (or an algebra $\underline{A}$ or a clone C) of the form

$\exists t_1, \dots, t_n$ terms $\mathcal{V} \models$ some identities with these terms

Examples

- $\exists p(x,y,z)$ $\quad \mathcal{V} \models p(x,x,y) \approx y \approx p(y,x,x)$

  (p: Mal'tsev term)

  ✓ $\mathcal{V}$ = groups $\quad p(x,y,z) = x \cdot y^{-1} \cdot z$

  ✓ $\mathcal{V}$ = quasigroups $\quad p(x,y,z) = x/(y\backslash y) \cdot y\backslash z$

  × $\mathcal{V}$ = semigroups, $\mathcal{V}$ = lattices

- $\exists m(x,y,z)$ $\quad \mathcal{V} \models m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x$

  (m: majority term)

  ✓ $\mathcal{V}$ = lattices $\quad m(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$

  × $\mathcal{V}$ = groups

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Why important?

- often Mal'tsev condition $\Leftrightarrow$ properties of relations
  $\Leftrightarrow$ nice properties of algebras

e.g.
- $\mathcal{V}$ congruence distributive
  nice → (recall Jónsson's lemma)
  $\Leftrightarrow$ $\mathcal{V}$ has "Jónsson terms"
  ($\Uparrow$ $\mathcal{V}$ has a majority term)
- $\mathcal{V}$ congruence modular
  $\Leftrightarrow$ $\mathcal{V}$ has "Gumm terms"
  ($\Uparrow$ $\mathcal{V}$ has a Mal'tsev term or a majority term)
  and "commutator theory works nicely

- an organizing principle in UA

e.g. strategy for solving problems
1. solve in groups, lattices
2. solve in algebras with a Mal'tsev term / majority
3. solve in algebras with Jónsson terms ....

⋮

(x) solve in general

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Mal'tsev's Mal'tsev condition

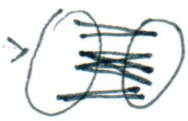
Def. $\underline{A}$ is CP (congruence permutable) if

$$\forall \alpha, \beta \in Con(\underline{A}) \quad \alpha \circ \beta = \beta \circ \alpha$$

Variety $\mathcal{V}$ is CP if $\forall \underline{A} \in \mathcal{V}$ $\underline{A}$ is CP

- $\underline{A}$, $\alpha, \beta \in Con(\underline{A})$. Then $\underline{A} \to \underline{A}/\alpha \times \underline{A}/\beta$ is iso
  $a \mapsto (a/\alpha, a/\beta)$

  iff $\alpha \wedge \beta = 0$, $\alpha \vee \beta = 1$, $\alpha \circ \beta = \beta \circ \alpha$

- $R \leq A \times B$ is <u>rectangular</u> if 

  $\forall a, a' \in A \ \forall b, b' \in B \ (a,b'), (a',b), (a',b') \in R \Rightarrow (a,b) \in R$

  $\Leftrightarrow \eta_A \circ \eta_B = \eta_B \circ \eta_A$ (where η's are the projection kernels)

Theorem (Mal'tsev '54) $\mathcal{V}$ variety

(i) $\mathcal{V}$ is CP

(ii) $\forall \underline{A}, \underline{B} \in \mathcal{V}$ $\forall R \leq \underline{A} \times \underline{B}$ R is rectangular

(iii) $\mathcal{V}$ has a Mal'tsev term

$$(\, p(x,x,y) \approx y \approx p(y,x,x))$$

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Thm (i) $\mathcal{V}$ is CP

(ii) $\forall \underline{A}, \underline{B} \in \mathcal{V}$ $\forall R \leq \underline{A} \times \underline{B}$ R is rectangular

(iii) $\mathcal{V}$ has a Mal'tsev term

Proof: (i) $\to$ (ii) from (ⅲ) above

(ii) $\to$ (iii)

- Take $\underline{A} = \underline{B} = \underline{F}_{\mathcal{V}}(\{x, y\}) \overset{\text{recall}}{=} F(\{x,y\})/\lambda_{\mathcal{V}}$ (recall)
- Take $R :=$ subuniverse of $\underline{A} \times \underline{B}$ generated by $\{(x,y), (x,x), (y,x)\}$ (formally, their $\lambda_{\mathcal{V}}$ classes)
- $(y,y) \in R$ (rectangularity)

$$(y,y) \overset{\text{def. of } R}{=} p^{\underline{A}\times\underline{B}}((x,y),(x,x),(y,x)) \quad \text{for some term } p$$

$$= (p^{\underline{A}}(x,x,y), p^{\underline{B}}(y,x,x))$$

$$= (p(x,x,y)/\lambda_{\mathcal{V}}, p(y,x,x)/\lambda_{\mathcal{V}})$$

- so $y/\lambda_{\mathcal{V}} = p(x,x,y)/\lambda_{\mathcal{V}} \Rightarrow \mathcal{V} \models p(x,x,y) \approx y$

$y/\lambda_{\mathcal{V}} = p(y,x,x)/\lambda_{\mathcal{V}} \Rightarrow \mathcal{V} \models p(y,x,x) \approx y$

(iii) $\to$ (i) $\underline{A} \in \mathcal{V}$ $\alpha, \beta \in \mathrm{Con}(\underline{A})$, $\alpha \circ \beta \overset{?}{\subseteq} \beta \circ \alpha$

- consider $(a,b) \in \alpha \circ \beta$ i.e. $\exists c$ $a \,\alpha\, c \,\beta\, b$
- then $p(a,c,c) \;\beta\; p(a,c,b) \;\alpha\; p(a,a,b)$

$\quad p(a,c,c) = a$, $\quad p(a,a,b) = b$

<u>Corollary</u> $\underline{A}$ finite TFAE

(i) $HSP(\underline{A})$ is CP

(ii) $\forall n\ \forall R \leq \underline{A}^n \times \underline{A}^n$ is rectangular ↗ info about compatible relations

(iii) $Clo(\underline{A})$ contains a Mal'tsev operation

Proof: recall $F_{HSP(\underline{A})}(\{x,y\}) \cong \underline{Clo_2 A} \leq \underline{A}^{A^2}$

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majority

• $R \subseteq A_1 \times A_2 \times A_3$ <u>2-decomposable</u> if $\forall a_1, a_2, a_3$

$(a_1, a_2, a_3) \in R$ iff $\exists b_1, b_2, b_3$ $(b_1, a_2, a_3) \in R$ & $(a_1, b_2, a_3) \in R$ & $(a_1, a_2, b_3) \in R$

Theorem [Baker, Pixley '70s] $\mathcal{V}$ variety TFAE

(i) $\forall \underline{A}_1, \underline{A}_2, \underline{A}_3 \in \mathcal{V}\ \forall R \leq \underline{A}_1 \times \underline{A}_2 \times \underline{A}_3$ 2-decomposable

(ii) $\mathcal{V}$ has a majority term ($m(xxy) \approx m(xyx) \approx m(yxx) \approx x$)

(ii) → (i) apply the term operation to [the three tuples above]

(i) → (ii) • $\underline{A}_1 = \underline{A}_2 = \underline{A}_3 = \underline{F_{\mathcal{V}}}(\{x,y\})$

• R .. subuniverse generated by $\{(y,x,x), (x,y,x), (x,x,y)\}$

• $(x,x,x) \in R$ (2-decomposability)

• the rest is as before