UA $\boxed{11.1}$

$\boxed{\text{RECAP}}$

<u>Clone on A</u> = set of operations on A closed under forming term operations

arity ≥ 1

= set of operations on A that

- contains all the projections π_i^n
- is closed under composition

$$f(g_1,\dots,g_m)(\bar a) = f(g_1(\bar a),\dots,g_m(\bar a))$$

$Clo(\underline{A})$ = the clone of term operations of $\underline{A}$

$Clo_n(\underline{A})$ = n-ary operations in $Clo(\underline{A})$

!!! $Clo_n(\underline{A}) \leq \underline{A}^{A^n}$ generated by $\pi_1^n,\dots,\pi_n^n$

because $f^{\underline{A}^{A^n}}(g_1,\dots,g_m) = \underbrace{f(g_1,\dots,g_m)}_{\text{composition}}$

basic operation in $\underline{A}^{A^n}$

THM: <u>$Clo_n(\underline{A})$</u> $\cong F_{\{\underline{A}\}}(\{x_1,\dots,x_n\})$

$\left(= F_{HSP(\underline{A})}(\{x_1,\dots,x_n\}) \right)$

(UA) ~~10.8~~ 11.2

COMPATIBILITY

<u>Def.</u> $f : A^n \to A$, $R \subseteq A^m$

f is <u>compatible</u> with R (or R invariant under f (compatible with f))

if $R \leq (A; f)$

Recall

$$\begin{array}{cccc} \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} & \begin{array}{c} \xmapsto{f} \\ \xmapsto{f} \end{array} & \begin{pmatrix} \\ \\ \\ \end{pmatrix} \end{array}$$

$$\begin{array}{cccccc} \in & \in & & \in & & \in \\ R & R & & R & \Longrightarrow & R \end{array}$$

(1) the set of all operations compatible with $R \subseteq A^m$ is a clone on A (see 10.2)

– some clones from the examples are of this sort

(2.) • f is compatible with $\{0\} \subseteq A^1$ iff ...

• f is compatible with $\{a\} \subseteq A^1$ $\forall a \in A$ iff ...

• f is —"— $B \subseteq A^1$ $\forall B \subseteq A$ iff ....

• f is —"— $R \subseteq \{0,1\}^2$ .. $R = \leq$

$(R = \{(0,0), (0,1), (1,1)\})$

iff ....

(2) Is $\wedge / \vee$ compatible with "$x \wedge y \to z$" ie. $R = \{0,1\}^3 \setminus \{(1,1,0)\}$?

UA | II.3

Pol–Inv: #2 Galois correspondence in UA

A – fixed

$Op :=$ all operations on A (arity ≥ 1)

$Rel :=$ all relations on A (nonempty)

relation $\Phi \subseteq Op \times Rel$ $\quad f \, \Phi \, R$ iff f compatible with R

sets of operations

$\mathcal{R}^{\leftarrow} =: \mathrm{Pol}(\mathcal{R}) =$ all operations compatible with every $R \in \mathcal{R}$

$\underline{A} \mapsto \underline{A}^{\rightarrow}$

sets of relations

$\underline{A}^{\rightarrow} =: \mathrm{Inv}(\underline{A}) =$ all relations invariant under every f in $\underline{A}$

$\mathcal{R} \mapsto \mathcal{R}^{\leftarrow}$

"unordered algebras"

$\}$ antiisomorphism

(closed sets of operations, $\subseteq$) $\longleftrightarrow$ (closed sets of relations, $\subseteq$)

$= \{\mathrm{Pol}(\mathcal{R});\ \mathcal{R} \subseteq Rel\}$ $\qquad = \{\mathrm{Inv}(\underline{A});\ \underline{A} \subseteq Op\}$

For finite A

closed sets of operations = clones

closed sets of relations = coclones

UA $\boxed{11.4}$

Remark

Notation $\mathrm{Inv}(\underline{A})$ also used for algebras

$\mathrm{Inv}(A; f_1, f_2, \dots) := \mathrm{Inv}(\{f_1, f_2, \dots\})$

<u>Note</u> $R \in \mathrm{Inv}(\underline{A})$ iff R is a subpower of $\underline{A}$

$\forall \mathcal{R} \subseteq \mathrm{Rel}$ $\quad \mathrm{Pol}(\mathcal{R})$ is a clone

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Proof: $\mathcal{R} = \{R_1, R_2, \dots\}$

$$\mathrm{Pol}(\mathcal{R}) = \{f;\ f \text{ compatible with } R_1\} \cap \{f;\ \text{———''———}\ R_2\} \cap \dots$$

clones

$\rightarrow$ every closed set of operations is a clone

Examples

- $\mathrm{Pol}(\emptyset)$ = all operations
- Pol(all relations) = projections $\quad$ Exercise
- $A = \{0,1\}$ $\mathrm{Pol}(\{\leq\})$ = monotone operations $(= \mathrm{Clo}(\{0,1\}; 0, 1, \wedge, \vee))$
- $\mathrm{Pol}(\{B;\ B \subseteq A\})$ = conservative operations
- $\mathrm{Pol}(\{\{a\};\ a \in A\})$ = idempotent operations

? $\mathrm{Pol}(\ ?\ ) = \mathrm{Clo}(\{0,1\}; \rightarrow)$

UA

$\boxed{11.5}$

closed ~~classes~~ sets of operations = clones

THEOREM (Geiger'68, Bodnarčuk, Kalužnin, Kotov, Romov'69)

For every <u>finite</u> algebra $\underline{A}$, $Clo(\underline{A}) = Pol(Inv(\underline{A}))$

Proof: $\subseteq$ ✓ (since $Pol(\ )$ is always a clone)

$\supseteq$
- take $f \in Pol(Inv(\underline{A}))$, say n-ary
- fix a linear ordering of A^n and recall

$$g \in A^{A^n} \longleftrightarrow \text{tuple } \tilde{g} \in A^{|A^n|}$$

$$R \subseteq A^{A^n} \longleftrightarrow \text{relation } \tilde{R} = \{\tilde{g} ; g \in R\} \subseteq A^{|A^n|}$$

- ~~Take~~ define $R := Clo_n(\underline{A})$
- recall $R \leq \underline{A}^{A^n} \Rightarrow \tilde{R} \leq \underline{A}^{|A^n|} \Rightarrow \tilde{R} \in Inv(\underline{A})$
- so f is compatible with $\tilde{R}$, in particular

$$|A^n| \left\{ \begin{array}{cccccc} \left(\tilde{\pi}^n_1 \right. & \tilde{\pi}^n_2 & \cdots & \left. \tilde{\pi}^n_n \right) & \overset{f}{\mapsto} \ \cdots & \tilde{f} \\ \in \tilde{R} & \in \tilde{R} & & \in \tilde{R} & \Rightarrow & \in \tilde{R} \end{array} \right. \Rightarrow f \in R \Downarrow f \in Clo_n(\underline{A})$$

---

so we have antiisomorphism

$$(\text{clones}, \subseteq) \longleftrightarrow (\text{closed sets of relations}, \subseteq)$$

? Is $p(x,y) = x+y$ in $Clo(A; \wedge, \vee)$?

UA 11.6

THE OTHER SIDE

<u>Def.</u> $R, R_1, \dots, R_n$ relations on A.

R is primitively positively definable (or pp-definable) from $R_1, \dots, R_n$ if R can be defined by a first order formula using

- $R_1, R_2, \dots, R_n, =$
- conjunction
- existential quantification

Example $R_1 \subseteq A$, $R_2 \subseteq A^2$, $R_3 \subseteq A^3$

$$R(x,y) \overset{\text{def}}{=} \underbrace{\exists u,v,w \; R_1(u) \wedge (x=v) \wedge R_2(v,y) \,\&\, R_3(y,x,x)}_{\text{pp-definition}}$$

What can we do

- intersection $R_1, \dots, R_n$ m-ary $\Rightarrow$ $R_1 \cap R_2 \cap \dots \cap R_n$ pp-definable from $R_1, \dots, R_n$
- permutation of coordinates R 3-ary $\Rightarrow$ $S = \{(a,b,c);\ (c,a,b) \in R\}$ pp-definable from R
- introduction of dummy coordinates R 2-ary $\Rightarrow$ $S = \{(a,b,c);\ (a,b) \in R\}$ ...
- projection R ternary $\Rightarrow$ $S = \{(b,c);\ \exists a\ (a,b,c) \in R\}$ pp. def. from R
- composition of binary relations

....

UA | 11.7

Def A <u>coclone</u> (or <u>relational clone</u>) on A is a set of relations on A that is closed under pp-definitions & contains $\emptyset$

For a set of relations $\mathbb{R}$

$$\text{Coclo}(\mathbb{R}) := \text{the smallest coclone containg } \mathbb{R}$$
$$= \text{all relations pp-definable from } \mathbb{R} + \emptyset$$

(since a relation pp-def. from relations pp-def. from $\mathbb{R}$ is pp-def. from $\mathbb{R}$)

6 $\text{Inv}(\underline{A})$ is a coclone (for each set of operations $\underline{A}$)

THEOREM For every set of relations $\mathbb{R}$ on a finite set A

$$\text{Inv}(\text{Pol}(\mathbb{R})) = \text{Coclone}(\mathbb{R})$$

so we have mutually inverse anti isomorphisms

$$\text{clones} \underset{\text{Pol}}{\overset{\text{Inv}}{\rightleftarrows}} \text{coclones}$$