UA $\boxed{10.1}$

WAS

I LATTICES

complete lattices, closure operators, Galois correspondences

II SEMANTICS

- H, S, P, varieties
- direct & subdirect decomposition

III SYNTAX

- terms, free algebras, identities
- Mod-Id Galois correspondence

NOW

IV CLONES & COCLONES

today:
- term operations vs. basic operations
- clones
- clones & free algebras

next week:
- Pol-Inv Galois correspondence

UA 10.2

n-element linearly ordered set

Def. $\underline{A}$ algebra, $X = \{x_1, \dots, x_n\}$, t term over X

Define $t^{\underline{A}}: A^n \to A$ naturally

(i.e., $t^{\underline{A}}(a_1, \dots, a_n) = \hat{m}(t)$, where $m(x_i) = a_i$)

$t^{\underline{A}}$ is called a term operation of $\underline{A}$

Example $\underline{A} = (A, \cdot)$, $X = \{x_1, x_2, x_3\}$, $t = (x_1 \cdot x_2) \cdot x_1$

$t^{\underline{A}}(a_1, a_2, a_3) = (a_1 \cdot a_2) \cdot a_1$

Note X needs to be linearly ordered

$\underline{A} \models s \approx t$ iff $s^{\underline{A}} = t^{\underline{A}}$

$R \leq \underline{A}^I$ ie. R is preserved by basic operations in $\underline{A}$

iff R — " — term operations in $\underline{A}$

$\to$ many properties (subuniverses, congruences) depend only on the set of term operations

also note $Sg_{\underline{A}}(B) = \{t^{\underline{A}}(b_1, \dots, b_n);\ n \in \mathbb{N}_0, b_1, \dots, b_n \in B, t \text{ term}\}$

most purposes $\to$ term operations > basic operations

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compare

set of permutations $\longrightarrow$ permutation group

set of mappings $A \to A$ $\longrightarrow$ transformation monoid

set of mappings $A^? \to A$ ("algebra") $\longrightarrow$ clone (or function clone)

UA 10.3

A set $\mathcal{C}$ of operations on A of arity ≥ 1

(formally $\mathcal{C} \subseteq \bigcup_{n=1,2,\dots} A^{A^n}$)

is a (function) clone if

Def 1 $\mathcal{C}$ is closed under forming term operations (in signature $\Sigma = \mathcal{C}$)

Ex If $f \in \mathcal{C}$ ternary and $g \in \mathcal{C}$ binary, then $h \in \mathcal{C}$ where $h(a_1, a_2, a_3, a_4) := f(g(a_1, a_2), a_1, a_4)$

Def 2 • $\mathcal{C}$ contains all the projections.. $\forall n \ \forall i \leq n \ \pi_i^n \in \mathcal{C}$, where $\pi_i^n(a_1, a_2, \dots, a_n) = a_i$

• $\mathcal{C}$ is closed under "composition":

If $f \in \mathcal{C}$ m-ary $g_1, \dots, g_m \in \mathcal{C}$ n-ary then $f(g_1, \dots, g_m) \in \mathcal{C}$, where

$$\boxed{f(g_1, \dots, g_m)(a_1, \dots, a_n) := f(g_1(a_1, \dots, a_n), \dots, g_m(a_1, \dots, a_n))}$$

Notation: $\mathcal{C}_n := \mathcal{C} \cap A^{A^n}$ n-ary members of $\mathcal{C}$

Note: only operations of arity ≥ 1 (technicality)

Def 1 $\Leftrightarrow$ Def 2

Note: always infinite

UA 10.4

Examples

- $\mathcal{C}$ = all operations on A
- ~~$\mathcal{C}$ = all unary operations on A~~ - not a clone
- $\mathcal{C}$ = all projections on A
- $\mathcal{C}$ = all conservative operations on A

  $f(a_1, \dots, a_n) \in \{a_1, a_2, \dots, a_n\}$ $(\forall a_1, \dots \in A)$
- $\mathcal{C}$ = all idempotent operations on A

  $f(a, \dots, a) = a$
- $\mathcal{C}$ = all monotone operations on A w.r.t. a partial order on A

  $a_1 \le b_1,\ a_2 \le b_2, \dots \Rightarrow f(a_1, \dots, a_n) \le f(b_1, \dots, b_n)$
- $\mathcal{C}$ = all polynomial functions on a ring $(A; \cdot, +, \dots)$

  (e.g. $f(x_1, x_2, x_3) = 2x_1^2x_2 + 7x_3^{137}$ on $\mathbb{Z}$)
- $\mathcal{C}$ = all linear functions of an R-module $(A; +, r\cdot, \dots)$

  (e.g. $f(x_1, x_2, x_3) = \frac{7}{3}x_1 + \frac{13}{7}x_2 + \frac{2}{5}x_3$

  for a $\mathbb{Q}$-module = vector space over $\mathbb{Q}$)
- $\mathcal{C}$ = all affine functions of an R-module

  (e.g. $f(x_1, x_2, x_3) = \frac{7}{3}x_1 + \frac{13}{7}x_2 + \frac{2}{5}x_3 + \frac{137}{31}$)

UA $\boxed{10.5}$

(all clones on A; $\subseteq$) is a complete algebraic lattice

$\wedge$ ... intersection

$\vee$ ... smallest clone containing $\cup$

- $|A| = 2$ countable & known **Post's lattice**
- $|A| > 2$ finite continuum many clones (Janov, Mučnik '59)

Def. $\underline{A}$ algebra, $Clo(\underline{A})$ = all term operations ~~on~~ of $\underline{A}$

$Clo_n(\underline{A})$ = all n-ary term operations of $\underline{A}$

$Clo(\underline{A})$ is a clone, every clone is of this form

If $Clo(\underline{A}) = Clo(\underline{B})$ we call $\underline{A}$ and $\underline{B}$ *term equivalent*

(?) $Clo(A;\ \text{no operations}) =$

$Clo(A; \wedge) =$ (where $\wedge$ is a semilattice operation)

$Clo(\{0,1\}; \wedge, \vee) =$

$Clo(\{0,1\}; \wedge, \neg) =$

$Clo(\mathbb{Z}_p; 1, +) =$

$Clo(\{0,1\};\ "x+y+z") =$

$Clo(\{0,1\};\ \text{NAND}) =$

UA 10.6

CLONES & FREE ALGEBRAS

useful viewpoint

$f \in A^{A^n}$ (n-ary operation) ~ tuple indexed by A^n

e.g.

	0	1	2
0	2	0	1
1	1	1	2
2	0	0	1

$$\sim \begin{array}{ccccccccc} 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \\ (2, & 0, & 1, & 1, & 1, & 2, & 0, & 0, & 1) \end{array}$$

i.e. $R \subseteq A^{A^n}$ (set of n-ary operations) ~ $|A^n|$-ary relation

!! $Clo_n(\underline{A}) \leq \underline{A}^{A^n}$ generated by projections $\pi_1^n, \pi_2^n, \ldots, \pi_n^n$ in $\underline{A}^{A^n}$ we have

Proof: for a basic operation $f^{\underline{A}}$ of $\underline{A}$

$$f^{\underline{A}^{A^n}}(g_1, \ldots, g_m) = \underbrace{f(g_1, \ldots, g_m)}_{\text{in DEF 2 of clone}}$$

say m-ary

$\boxed{\text{THEOREM}}$ $Clo_n(\underline{A}) \simeq F_{\{\underline{A}\}}(\{x_1, \ldots, x_n\})$

(Recall $= F_{HSP(\underline{A})}(\{x_1, \ldots, x_n\})$)

Proof: $\theta : F_{\{A\}}(x_1, \ldots, x_n) \longrightarrow Clo_n(\underline{A})$ is an isomorphism

$$t/\lambda_{\{A\}} \longmapsto t^{\underline{A}}$$

- well defined ($t \lambda_{\underline{A}} s \Rightarrow \underline{A} \models t \approx s \Rightarrow t^{\underline{A}} = s^{\underline{A}}$)
- homomorphism
- onto
- one-to-one ($t^{\underline{A}} = s^{\underline{A}} \Rightarrow \underline{A} \models t \approx s \Rightarrow t/\lambda_{\{A\}} = s/\lambda_{\{A\}}$

(UA) $\boxed{10.7}$

- $Clo_n(\underline{A})$ is the subuniverse of $\underline{A}^{A^n}$ generated by the n-ary projections
- $\underline{Clo_n}(\underline{A})$ is isomorphic to the free algebra for $\{\underline{A}\}$ (or $HSP(\underline{A})$) over n-element set of variables

$\rightsquigarrow$ a way to compute free algebras

Example

$\mathcal{V}$ = distributive lattices $= HSP(\{0,1\}; \wedge, \vee)$

$F_{\mathcal{V}}(x_1,\dots,x_n)$ = terms over $\{x_1,\dots,x_n\}$ modulo identities

$\overset{\text{THEOREM}}{\cong} \underline{Clo_n}(\{0,1\}; \wedge, \vee)$

$= ($n-ary monotone idempontent operations; natural $\wedge, \vee)$

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say, we do not know) ; say $n=2$

	00	01	10	11	
π^2_1	(0,	0,	1,	1)	
π^2_2	(0,	1,	0,	1)	
	(0,	0,	0,	1)	($\wedge$ of 1st & 2nd tuple)
	(0,	1,	1,	1)	($\vee$ —〃—)

already closed $\to$ we have the free algebra

UA 10.8

COMPATIBILITY

Def. $f: A^n \to A$, $R \subseteq A^m$

f is <u>compatible</u> with R (or R invariant under f (compatible with f))

if $R \leq (A; f)$

Recall

$$\begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \begin{array}{c} \stackrel{f}{\mapsto} \\ \stackrel{f}{\mapsto} \end{array} \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

$\begin{array}{ccccc} \Uparrow & \Uparrow & \Uparrow & & \Uparrow \\ R & R & R & \Longrightarrow & R \end{array}$

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in ⓪ the set of all operations compatible with $R \subseteq A^m$ is a clone on A (see 10.2)

– some clones from the examples are of this sort

② • f is compatible with $\{0\} \subseteq A'$ iff ...

• f is compatible with $\{a\} \subseteq A'$ $\forall a \in A$ iff ...

• f is —"— $B \subseteq A'$ $\forall B \subseteq A$ iff ....

• f is —"— $R \subseteq \{0,1\}^2$ .. $R = \leq$
($R = \{(0,0), (0,1), (1,1)\}$)

iff ....

② Is $\wedge / \vee$ compatible with "$x \wedge y \to z$" ie. $R = \{0,1\}^3 \setminus \{(1,1,0)\}$?