UA $\boxed{9.1}$

$\boxed{\text{RECAP}}$

Absolutely free algebra over X

$$\underline{F}(X) = (\text{ terms over } X;\ \text{natural operations})$$

Free algebra for $\mathcal{K}$ over X

$$\underline{F}_{\mathcal{K}}(X) = \underline{F}(X)/\lambda_{\mathcal{K}}$$

$$\lambda_{\mathcal{K}} = \bigwedge \{\alpha \in \operatorname{Con} \underline{F}(X);\ \underline{F}(X)/\alpha \in S(\mathcal{K})\}$$

$\rightarrow$ $\underline{F}_{\mathcal{K}}(X) \in SP(\mathcal{K})$

@ (i) $s \; \lambda_{\mathcal{K}} \; t$

(ii) $\mathcal{K} \models s \approx t$

(iii) $HSP(\mathcal{K}) \models s \approx t$

(iv) $F_{\mathcal{K}}(X) \models s \approx t$

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useful ☺ $\forall m : X \to A \;\; \exists !\ \underline{F}(X) \to \underline{A}$ extending m,
namely $\hat{m}$

UA $\boxed{9.2}$

$\boxed{\text{EXAMPLE}}$

write xx, $xy\cdot z \approx zy \cdot x$

$$\mathcal{K} = \{(A,\cdot);\ (A,\cdot) \models \{x\cdot x,\ (x\cdot y)\cdot z \approx (z\cdot y)\cdot x\}\}$$

$y\cdot xy \approx x$

what is $\underline{F} := \underline{F}_{\mathcal{K}}(x,y)$?

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playing with identities $\rightsquigarrow$ each term is equivalent (modulo $\lambda_{\mathcal{K}}$) to one of $\{x, y, xy, yx\}$ (use (ii))

candidate $\underline{B} := \{\underbrace{x,\ y,\ xy,\ yx}_{\text{4 distinct}}\ ;\ *\}$

$*$	x	y	xy	yx
x	x	xy	yx	y
y	yx	y	x	xy
xy	y	yx	xy	x
yx	xy	x	y	yx

Is $\underline{B} \cong \underline{F}$? Not clear, e.g. why $xy \not\approx^{\lambda_{\mathcal{K}}} x$ ($\mathcal{K} \not\models xy \approx x$)

Enough to show $\underline{B}$ satisfies $uu \approx u$, $uv\cdot w \approx wv\cdot u$ (for all $u, v, w \in B$!)

$*$	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

$*$	00	11	01	10
00	00	01	10	11
11	10	11	00	01
01	11	10	01	00
10	01	00	11	10

$(u_1,u_2) * (v_1,v_2) =$

$= (u_2 + v_1 + v_2,\ u_1 + u_2 + v_1)$

$\underline{B} \cong (\mathbb{Z}_2 \times \mathbb{Z}_2, *)$ $\quad \vec{u} * \vec{v} = A\vec{u} + A^2\vec{v}$ $\quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

verification now easy

UA 9.3

Mod – Id : #1 Galois correspondence in UA

Σ – still fixed

Alg := all algebras in signature Σ

Ident := all identities in Σ over $X = \{x_1, x_2, \dots\dots\}$

relation $\Phi \subseteq \mathrm{Alg} \times \mathrm{Ident}$ $\quad \underline{A} \,\Phi\, s \approx t$ iff $\underline{A} \models s \approx t$

classes of algebras

$\mathcal{E}^{\leftarrow} =: \mathrm{Mod}(\mathcal{E}) = \{\underline{A};\ \underline{A} \models \mathcal{E}\}$

all algebras satisfying $\mathcal{E}$

$\mathcal{K} \mapsto \mathcal{K}^{\rightarrow}$

sets of identities

$\mathcal{K}^{\rightarrow} =: \mathrm{Id}(\mathcal{K}) = \{s \approx t;\ \mathcal{K} \models s \approx t\}$

all identities satisfied in $\mathcal{K}$

$\mathcal{E} \mapsto \mathcal{E}^{\leftarrow}$

antiisomorphisms

(closed classes of algebras, $\subseteq$) $\longleftrightarrow$ (closed classes of identities, $\subseteq$)

$= \{\mathrm{Mod}(\mathcal{E});\ \mathcal{E} \subseteq \mathrm{Ident}\}$ $\qquad = \{\mathrm{Id}(\mathcal{K});\ \mathcal{K} \subseteq \mathrm{Alg}\}$

!!! closed classes of algebras = varieties

! closed classes of identities = equational theories

"variety = class of algebras defined by identities"

$\Leftarrow$ we know

$\Rightarrow$ we now prove

UA $\boxed{9.4}$

THEOREM (Birkhoff '35) For every class of algebras $\mathcal{K}$

$$HSP(\mathcal{K}) = Mod(Id(\mathcal{K}))$$

proof: $\subseteq$ ✓

$\supseteq$ • take $\underline{A} \in Mod(Id(\mathcal{K}))$

$\underline{A}$ satisfies all the identities satisfied in $\mathcal{K}$

• define $\beta = \ker \widehat{id} \in Con(\underline{F}(A))$

(ie $s \, \beta \, t$ iff $\widehat{id}(s) = \widehat{id}(t)$)

$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, phantom, "\cap"] & \underline{A} \\ \underline{F}(A) \arrow[ur, "\widehat{id}"'] & \end{tikzcd}$$

• note $\underline{F}(A)/\beta \cong \underline{A}$ (1st iso. t.)

!• $\lambda_{\mathcal{K}} \leq \beta$: $s \, \lambda_{\mathcal{K}} \, t \overset{thm.}{\Rightarrow} \mathcal{K} \models s \approx t$

$\Rightarrow \underline{A} \models s \approx t \overset{def. of \beta}{\Longrightarrow} s \, \beta \, t$

$$\underline{A} \cong \underline{F}(A)/\beta \overset{2nd\ iso}{\cong} \underline{F}(A)/\lambda_{\mathcal{K}} \Big/ \beta/\lambda_{\mathcal{K}} \in HSP(\mathcal{K})$$

$= F_{\mathcal{K}}(A) \in SP(\mathcal{K})$ PROP.

• so we have antiisomorphism (varieties, $\subseteq$) $\leftrightarrow$ (closed cl. of id, $\subseteq$)

• sometimes easy to show $\mathcal{K}$ is a variety, hard to find $\mathcal{E}$ such that $\mathcal{K} = Mod(\mathcal{E})$

Ex. $\mathcal{V}, \mathcal{W}$ varieties of groups

$\mathcal{V} \circ \mathcal{W} := \{ \underline{G} ; \exists \underline{N} \trianglelefteq \underline{G} \ \underline{N} \in \mathcal{V}, \ G/N \in \mathcal{W} \}$ is a variety

$\mathcal{V} = \mathcal{W} =$ abelian groups $\quad \mathcal{V} \circ \mathcal{W} =$ 2-step solvable

UA $\boxed{9.5}$

FINITE BASE

variety $\mathcal{K}$ is <u>finitely based</u> if $\mathcal{K} = \mathrm{Mod}(\mathcal{E})$ for some finite $\mathcal{E}$

say $\mathcal{K} = HSP(\underline{A})$, $\underline{A}$ finite with finite Σ

- not necessarily finitely based
- no algorithm to check whether ~~there is a~~ it is (McKenzie '96)

Examples

- finitely based: finite groups (Oates, Powell '64)

$$HSP(\underline{D}_8) = HSP(Q_8) = \mathrm{Mod}(x^4 \approx 1,\ x^2 y \approx x y^2,\ \text{group axioms})$$

dihedral — quaternion

- non-finitely based
  - a 7-element algebra with a binary op. (Lyndon '54)
  - a 3-element — " — (Murski '65)

	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

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THE OTHER SIDE

$\mathcal{E}$ ... set of identities

closure of $\mathcal{E}$ = $\mathrm{Id}(\mathrm{Mod}\,\mathcal{E})$ ... all identities that ~~syntactically follow~~ semantically follow from $\mathcal{E}$

Def. $\mathcal{E} \models s \approx t$ if $\forall \underline{A}\ \ \underline{A} \models \mathcal{E} \Rightarrow \underline{A} \models s \approx t$

$s \approx t$ semantically follows from $\mathcal{E}$

.... would be nice if ....

closure of $\mathcal{E}$ = all identities that syntactically follow from $\mathcal{E}$

we wish

? **Def.** $\mathcal{E} \vdash s \approx t$ if $s \approx t$ can be obtained from $\mathcal{E}$ by "such and such" syntactic manipulations

$s \approx t$ syntactically follows from $\mathcal{E}$

? **Thm** $\mathcal{E} \models s \approx t$ iff $\mathcal{E} \vdash s \approx t$

Todos

- what is "such and such"?
- prove $\Rightarrow$ in Thm

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Example

$\mathcal{E} = \{\text{group axioms}, x^2 \approx 1\} \models xy \approx yx$

says every group satisfying $x^2 \approx 1$ is commutative

it can be shown by syntactic manipulations:

$xy \approx xyxx \approx xyxyyx \approx yx$

closer look

$xy \approx xy \cdot 1 \approx xy \cdot xx \approx \dots$

$z \approx z \cdot 1$
applied to $z = xy$
substitution

$1 \approx xx$
$xy \approx xy$
$\downarrow$
$xy \cdot 1 \approx xy \cdot xx$
applying operations to identities

---

we have used

- $\approx$ is an equivalence on $F(X)$
- invariant under applying operations

$\Leftrightarrow$ $\approx$ is a congruence of $\underline{F}(x)$

- invariant under substitutions

substitution = endomorphism of $\underline{F}(X)$

invariant under substitutions
= fully invariant: $s \approx t \Rightarrow e(s) \approx e(t)$ $\forall e \in End(\underline{F}(X))$

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we have observed: $\approx$ is a fully invariant congruence of $\underline{F}(X)$ $\quad (\approx = \mathrm{Id}(\mathcal{X}))$

Is it enough? YES!

Def. equational theory = fully invariant congruence of $\underline{F}(X)$

$\mathcal{E} \vdash s \approx t$ if (s,t) is in the equational theory generated by $\mathcal{E}$

(can be described in a more concrete way...)

Theorem $\mathcal{E} \models s \approx t$ iff $\mathcal{E} \vdash s \approx t$

($\Rightarrow$ $\mathrm{Id}(\mathrm{Mod}(\mathcal{E}))$ = equational theory generated by $\mathcal{E}$)

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so we have anti isomorphisms (mutually inverse)

$$\text{varieties} \underset{\mathrm{Mod}}{\overset{\mathrm{Id}}{\longleftrightarrow}} \text{equational theories}$$

Proof: nice exercise (not required for exam)