UA $\boxed{8.1}$

WAS

- LATTICES
  complete lattices, closure operators, Galois correspondences
- SEMANTICS
  - H, S, P, varities
  - direct & subdirect decompositions

NOW

- SYNTAX
  - terms, free algebras, identities
  - Mod-Id Galois correspondence

Σ - fixed signature

UA $\boxed{8.2}$

$\boxed{\text{TERMS}}$

"Def" Σ signature, X set (of variables)

<u>Σ-term</u> (term in signature Σ) <u>over X</u> is

a formal meaningful expression formed by

- variables in X (not all need to be used)
- symbols in Σ
- composition

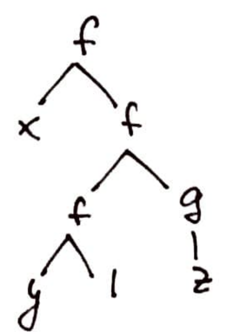
<u>Example</u> $\Sigma = \{ f, g, 1 \}$ $ar(f) = 2$, $ar(g) = 1$, $ar(1) = 0$

$X = \{ x, y, z \}$

non-terms $f(x, y, z)$, $f(x, ($, ...

terms x, 1, $f(x, x)$, $f(x, f(f(y, 1), g(z)))$

often drawn as trees



term of <u>height</u> 3

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formal definitions....

UA 8.3

"Def" $\underline{A}$ algebra in Σ, t Σ-term over X, $m: X \longrightarrow A$

<u>m-evaluation of t in $\underline{A}$</u> is denoted $\hat{m}(t) \in A$

(the meaning should be obvious)

<u>Ex</u> $t = f(x, f(f(y,1), g(z)))$

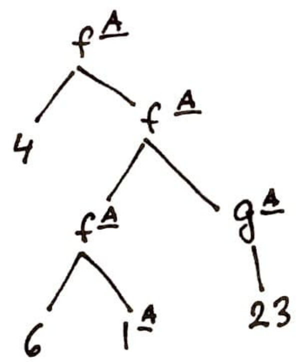
$m: X \to A$

$x \mapsto 4$

$y \mapsto 6$

$z \mapsto 23$

$\hat{m}(t) =$



$\hat{m}(t) = f^{\underline{A}}(4, f^{\underline{A}}(f^{\underline{A}}(6, 1^{\underline{A}}), g^{\underline{A}}(23)))$

<u>Def.</u> • $(\Sigma$-$)$<u>identity</u> : ordered pair (s,t) of terms in Σ, written $s \approx t$

• $\underline{A}$ algebra (in sig. Σ). $\underline{A}$ <u>satisfies</u> $s \approx t$ if $\forall m: X \to A$ $\hat{m}(s) = \hat{m}(t)$, written $\underline{A} \models s \approx t$

• $\mathcal{K}$ class of algebras (in Σ), $\mathcal{E}$ set of identities
$\mathcal{K}$ <u>satisfies</u> $\mathcal{E}$, written $\mathcal{K} \models \mathcal{E}$, if $\forall \underline{A} \in \mathcal{K}\ \forall s \approx t \in \mathcal{E}\ \underline{A} \models s \approx t$

<u>Ex.</u> • every group satisfies $\{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x \cdot 1 \approx x\}$

• $\underline{A} \models x \approx y$ iff $\underline{A}$ is 1-element

!!! ① • $\mathcal{E}$ ... set of identities $\quad \mathcal{K} \models \mathcal{E} \Rightarrow HSP(\mathcal{K}) \models \mathcal{E}$

($\Leftarrow$ is triv.)

$\Rightarrow \{\underline{A} ; \underline{A} \models \mathcal{E}\}$ is a variety

UA $\boxed{8.4}$

ABSOLUTELY FREE ALGEBRA

"something out of nothing"

Σ signature $\quad X$ set

$\underline{F}_\Sigma(X)$ or $\underline{F}(X)$ <u>absolutely free algebra</u> over X in sig. Σ

signature: Σ
universe: $F(X) =$ all Σ-terms over X
operations: naturally

<u>Examples</u> • $\Sigma = \{\bullet \text{ binary}\}$, $X = \{x, y, z\}$

$F(X) = \{x, y, z, x\cdot y, y\cdot x, \dots, (x\cdot z)\cdot(y\cdot(y\cdot x)), \dots\}$

$$\underbrace{(x\cdot y)\cdot z}_{\text{element of } F(X)} \bullet^{\underline{F}(X)} \underbrace{(x\cdot x)} := ((x\cdot y)\cdot z)\cdot(x\cdot x)$$

• $\Sigma = \{f \text{ unary}\}$ $X = \{x\}$

$F(\Sigma) = \{x, f(x), \underbrace{f(f(x))}_{f^{(2)}(x)}, \dots\}$

$$f^{\underline{F}(\Sigma)}(f^{(n)}(x)) = f^{(n+1)}(x)$$

$F(X) \cong ?$

X generates $\underline{F}(X)$

$\forall \underline{A}$ alg. $\forall m: X \to A$ $\exists!$ homo $\underline{F}(X) \to \underline{A}$ extending m, namely $\hat{m}$

since • $\hat{m}$ is a homomorphism
• $\hat{m}$ extends m
• homo determined on generators

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, phantom, "\cap|"] & \underline{A} \\ \underline{F}(X) \arrow[ur, "\hat{m}"'] & \end{tikzcd}$$

UA 8.5

FREE ALGEBRA FOR A CLASS

$\mathcal{K}$ ... class of algebras (in sig. Σ), X set

def. $\lambda_{\mathcal{K}} \in \operatorname{Con}(\underline{E}(X))$ by

$$\lambda_{\mathcal{K}} := \bigwedge \{\alpha \in \operatorname{Con}(\underline{E}(X));\ \underline{E}(X)/\alpha \in S(\mathcal{K})\}$$

$$F_{\mathcal{K}}(X) := \underline{E}(X)/\lambda_{\mathcal{K}}$$

<u>free algebra for $\mathcal{K}$ over X</u>

Proposition $F_{\mathcal{K}}(X) \in SP(\mathcal{K})$

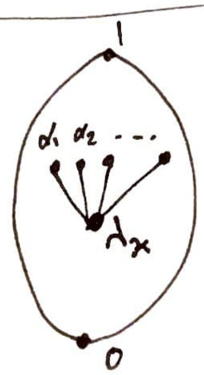
Proof:
- $\lambda_{\mathcal{K}} = \bigwedge_{i \in I} \alpha_i$ from the definition ($\underline{E}(X)/\alpha_i \in S(\mathcal{K})$)
- in $\operatorname{Con} \underline{F}_{\mathcal{K}}(X) = \operatorname{Con} \underline{E}(X)/\lambda_{\mathcal{K}}$ $\quad \bigwedge_{I} \alpha_i/\lambda_{\mathcal{K}} = 0$
- thm. on subdirect decompositions $\Rightarrow$

  $F_{\mathcal{K}}(X)$ is isomorphic to a subdirect product of

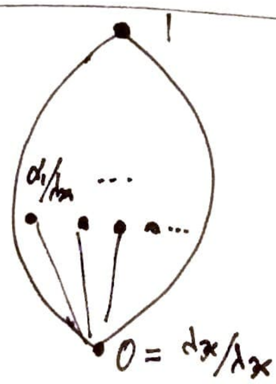
  $$\underline{F}_{\mathcal{K}}(X)/_{\alpha_i/\lambda_{\mathcal{K}}} \overset{\text{2nd iso thm.}}{\cong} F(\underline{X})/\alpha_i \in S(\mathcal{K})$$

- $\underline{F}_{\mathcal{K}}(X) \in SPS(\mathcal{K}) \overset{\text{"PS} \subseteq \text{SP"}}{\subseteq} SP(\mathcal{K})$

$\operatorname{Con} \underline{E}(X)$



$\operatorname{Con} \underline{E}(X)/\lambda_{\mathcal{K}}$



UA $\boxed{8.6}$

$\Longrightarrow F_{\mathcal{K}}(X) \in \mathcal{K}$ if $\mathcal{K}$ is a variety

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$\boxed{\text{THEOREM}}$ $\mathcal{K}$ class of algebras, X set, $s, t \in F(X)$ @

(i) $s \; \lambda_{\mathcal{K}} \; t$

(ii) $\mathcal{K} \models s \approx t$

} "$F_{\mathcal{K}}(X)$ = terms modulo identities satisfied in $\mathcal{K}$"

(iii) $HSP(\mathcal{K}) \models s \approx t$

(iv) $\underline{F_{\mathcal{K}}}(X) \models s \approx t$

} "identities true in $\mathcal{K}$ = — " — $HSP(\mathcal{K})$ = — " — $\underline{F_{\mathcal{K}}}(X)$"

Proof $(i) \Rightarrow (ii)$

- take $\underline{A} \in \mathcal{K}$, $m: X \to A$ want $\hat{m}(s) = \hat{m}(t)$
- $\hat{m}: \underline{F}(X) \to \underline{A}$ is a homomorphism
- $\underline{F}(X)/\ker \hat{m} \overset{\text{1st Iso}}{\cong} \hat{m}(F(X)) \in S(\mathcal{K})$
- $\lambda_{\mathcal{K}} \leq \ker \hat{m}$ (by def. of $\lambda_{\mathcal{K}}$)
  $s \; \lambda_{\mathcal{K}} \; t$ (assumption) } $s \ker \hat{m} \; t \Rightarrow \hat{m}(s) = \hat{m}(t)$

$(ii) \Rightarrow (iii)$ ✓ (☺)

$(iii) \Rightarrow (iv)$ follows from $F_{\mathcal{K}}(X) \in HSP(\mathcal{K})$ (Proposition)

$(iv) \Rightarrow (i)$

- the quotient homomorphism $q_{\lambda_{\mathcal{K}}}: \underline{F}(X) \to \underline{F}(X)/\lambda_{\mathcal{K}}$
  $x \longmapsto x/\lambda_{\mathcal{K}}$

  is equal to $\hat{m}$ for some $m: X \to F(X)/\lambda_{\mathcal{K}}$ (which?)

- $\underline{F}(X)/\lambda_{\mathcal{K}} = F_{\mathcal{K}}(X) \models s \approx t \Rightarrow \hat{m}(s) = \hat{m}(t) \Rightarrow q(s) = q(t)$
  $\Rightarrow s \; \lambda_{\mathcal{K}} \; t$

UA $\boxed{8.7}$

Thm $\textcircled{a}$ (i) $s \, d_{\mathcal{K}} \, t$

(ii) $\mathcal{K} \models s \approx t$

(iii) $\cancel{\mathcal{K}}$ $HSP(\mathcal{K}) \models s \approx t$

(iv) $\underline{F}_{\mathcal{K}}(X) \models s \approx t$

$\textcircled{o}$ $\underline{F}_{\mathcal{K}}(X) = \underline{F}_{HSP(\mathcal{K})}(X)$ (from (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii))

$\textcircled{o}$ $\underline{A} \in HSP(\mathcal{K})$, $m: X \to A \Rightarrow \exists!$ homo. $\underline{F}_{\mathcal{K}}(X) \to \underline{A}$ extending m

"categorical definition of $F_{\mathcal{K}}(X)$" (Proof: exercise)

$\textcircled{o}$ $\underline{A} \in HSP(\mathcal{K})$, $|X| \geq |A| \Rightarrow \underline{A}$ is isomorphic to a quotient of $F_{\mathcal{K}}(\underline{A})$ (from $\textcircled{o}\uparrow$)

" each algebra in $HSP(\mathcal{K})$ is ~~iso~~ a homomorphic image of the free algebra for $\mathcal{K}$ over sufficiently big X "

$\textcircled{o}$ $\mathcal{K}$ nontrivial $\Rightarrow d_{\mathcal{K}} \restriction X = 0$ ... " $X \subseteq$ " $F_{\mathcal{K}}(X)$

(ie $x, y \in X$, $x \neq y \Rightarrow x/d_{\mathcal{K}} \neq y/d_{\mathcal{K}}$)

Ex. • $\mathcal{V} = \{(A, \overset{\text{unary}}{f});\ f^3(x) \approx x\}$

– $F_{\mathcal{V}}(\{x\}) = \{x/d_{\mathcal{V}},\ f(x)/d_{\mathcal{V}},\ f^2(x)/d_{\mathcal{V}})$

– (!) $x/d_{\mathcal{V}}$, $f(x)/d_{\mathcal{V}}$, $f^2(x)/d_{\mathcal{V}}$ pairwise different (why?)

$\underline{F}_{\mathcal{V}}(\{x\}) \cong (\{0, 1, 2\},$ succ mod 3$)$

(UA) $\boxed{8.8}$

- $\mathcal{V} = \{\text{all abelian groups}\}$ $X = \{x_1, \dots, x_n\}$

  denote eg. $-2x_1 + 3x_2 := (((-x_1 + (-x_1)) + x_2) + x_2) + x_2$

  $\mathcal{V} \models (((-x_1 + x_2) + (-x_1)) + x_2) + x_2 \approx -2x_1 + 3x_2$

  – each term $\approx$ (in $\mathcal{V}$) $k_1x_1 + \dots + k_nx_n$ for some $k_i \in \mathbb{Z}$

  – (!) if $(k_1, \dots, k_n) \neq (l_1, \dots, l_n)$, then in $\mathcal{V}$ $k_1x_1 + \dots + k_nx_n \not\approx l_1x_1 + \dots + l_nx_n$ (since $(\mathbb{Z}, +, -, 0)$ doesn't satisfy it)

  – $(k_1x_1 + \dots + k_nx_n) + (l_1x_1 + \dots + l_nx_n) \approx (k_1 + l_1)x_1 + \dots + (k_n + l_n)x_n$

  $\rightsquigarrow$ $\underline{F}_{\mathcal{V}}(X) \cong (\{k_1x_1 + \dots + k_nx_n;\ k_i \in \mathbb{Z}\}, \text{natural op.}) \simeq \mathbb{Z}^n$

- $\mathcal{V} = \{\text{all semigroups}\}$ $X = \{x_1, \dots, x_n\}$

  $F_{\mathcal{V}}(X) \cong (\{\text{nonempty words over } X\}, \text{concatenation})$

- $\mathcal{V} = \{\text{all semilattices}\}$ $X = \{x_1, \dots, x_n\}$

  $\underline{F}_{\mathcal{V}}(X) \cong (P_{\text{nonempty}}(X); \cup)$

- $\mathcal{V}$ = lattices - a book "Free Lattices"

- $\mathcal{V}$ = commutative rings with 1 $F_{\mathcal{V}}(\{x\}) = \mathbb{Z}[x]$

- $\mathcal{V}$ = groups satisfying $x^n = 1$ for a fixed n

  is $F_{\mathcal{V}}(X)$ finite for each finite X? "Bounded Burnside Problem"

  ✓ $n = 2$ (abelian)

  ✓ $n = 3, 4, 6$ (hard)

  ? $n = 5$

  × for some n (Adian, Novikov)