UA 7.1

RECAP

DIRECT DECOMPOSITION

$$\left.\begin{array}{l}\alpha,\beta \in \operatorname{Con} \underline{A} \\ \alpha \wedge \beta = 0_A \\ \alpha \circ \beta = 1_A\end{array}\right\} \quad \underline{A} \cong \underline{A}/\alpha \times \underline{A}/\beta$$

$$a \mapsto (a/\alpha,\ a/\beta)$$

- every decomposition is such
- nontrivial $\Leftrightarrow$ $\alpha \neq 0,\ \beta \neq 0$

SUBDIRECT DECOMPOSITION

$R \leq_{sd} \prod_I \underline{A}_i$ if $\forall i\ \pi_i(R) = A_i$

$$\left.\begin{array}{l}\alpha_i \in \operatorname{Con} \underline{A},\ i \in I \\ \bigwedge \alpha_i = 0_A\end{array}\right\} \quad \underline{A} \simeq \underline{B} \leq_{sd} \prod \underline{A}/\alpha_i$$

$$a \longmapsto (a/\alpha_i)_{i \in I}$$

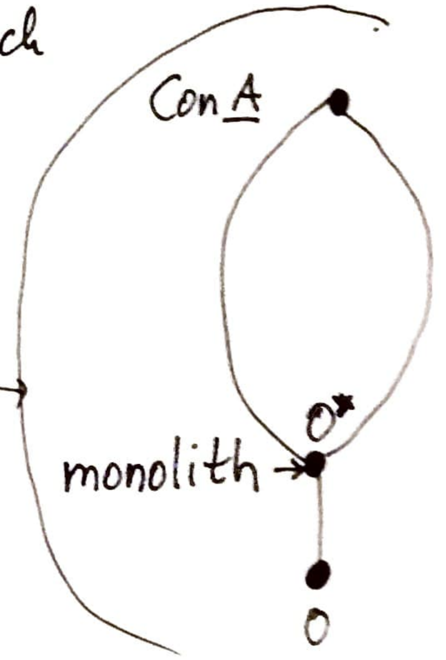
- every decomposition is such
- nontrivial $\Leftrightarrow$ $\forall i\ \alpha_i \neq 0$

$\underline{A}$ is SI

$\Leftrightarrow$ 0_A is completely $\wedge$-irreducible

$\Leftrightarrow$ $\exists a \neq b \in A\ \ \forall \alpha \in \operatorname{Con} \underline{A}$

$\alpha \neq 0 \Rightarrow a\ \alpha\ b$



THEOREM (Birkhoff '44)

Every algebra is isomorphic to a subdirect product of SIs (= SI algebras)

Proof: • for $a,b \in A$, $a \neq b$ take

$\theta_{a,b}$:= a maximal congruence not containing (a,b) (exists by Zorn)

☺ $\theta_{a,b}$ is completely $\wedge$-irreducible (the monolith is $\theta_{a,b} \vee Cg(a,b)$)

☺ $\bigwedge_{a \neq b} \theta_{a,b} = 0_A$ (by definition of $\theta_{a,b}$)

• $\underline{A} \cong \underline{B} \leq_{sd} \prod_{a \neq b} \underline{A}/\theta_{a,b}$ ⤳ SI since

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THEOREM: ... finite ... finite subdir. prod. of finite SIs

$SI(\mathcal{V})$ ... SIs in $\mathcal{V}$

Note: SP way better than HSP

☺ • $\mathcal{V}$ variety $\Rightarrow \mathcal{V} = SP(SI(\mathcal{V}))$ $(= HSP(SI(\mathcal{V})))$

• $\mathcal{V} \subseteq \mathcal{W} \iff \mathcal{V}_{SI} = \mathcal{W}_{SI}$

"variety determined by SIs!"

⤳ finding subvarieties, identity checking, ...

UA 7.3

$\mathcal{V}$ = all distributive lattices

- the only SI is  (and • ...)

$\Rightarrow$ $\mathcal{V} = SP($$)$ $\Rightarrow$ every distributive lattice is isomorphic to a sublattice of $(P(X), \cap, \cup)$

(= HSP( ))

it is a minimal variety

$\Rightarrow$ $\mathcal{V}$ has only two subvarieties: trivial, $\mathcal{V}$

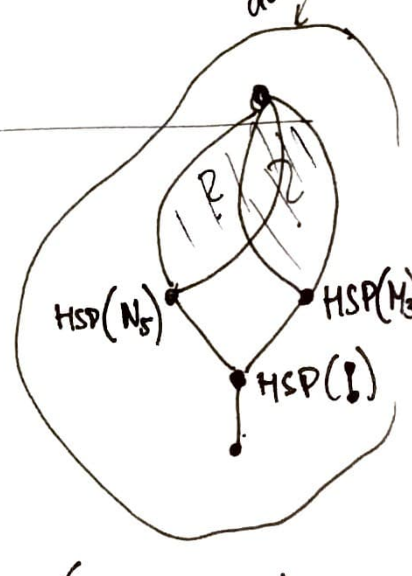
$\Rightarrow$ the lattice of varieties of lattices is

since $\mathcal{W} \subseteq \mathcal{V} \Leftrightarrow \mathcal{W}_{SI} \subseteq \mathcal{V}_{SI}$
and $\mathcal{W} = (H)SP(\mathcal{W}_{SI})$

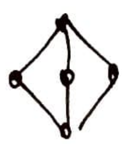
$\Rightarrow$ an identity $t \approx s$ is satisfied by every distributive lattice

iff it is satisfied by 

actually



$\mathcal{V}$ = all lattices

- many SIs, e.g.  

- SIs in Abelian groups .. $\mathbb{Z}_{p^k}$, $\mathbb{Z}_{p^\infty}$ (Pf ~ 1 page)

UA 7.4

THEOREM (a special case of "Jónsson's Lemma" '67)

If • $\mathcal{V} = HSP(\underline{A}_1, \dots, \underline{A}_n)$ all $\underline{A}_i$'s finite ← finitely generated

• $\forall \underline{B} \in \mathcal{V}$ Con $\underline{B}$ is distributive ← congruence distributive

Then $SI(\mathcal{V}) \subseteq HS(\underline{A}_1, \dots, \underline{A}_n)$

Example: lattices are congruence distributive (will see later)

Application: subvarieties of HSP()

• HS() = 

• if $\mathcal{W} \subseteq$ HSP(), then

$SI(\mathcal{W}) \subseteq SI(\quad) \overset{THM}{\subseteq}$ 

• $\Rightarrow \mathcal{W} = SP(\mathcal{K})$ $(= HSP(\mathcal{K}))$ for some $\mathcal{K}$

• $\Rightarrow \mathcal{W} \in \{$ triv, distributive, $HSP(N_5)\}$

since • if $\mathcal{K}$ contains  then $\mathcal{W} = HSP(N_5)$

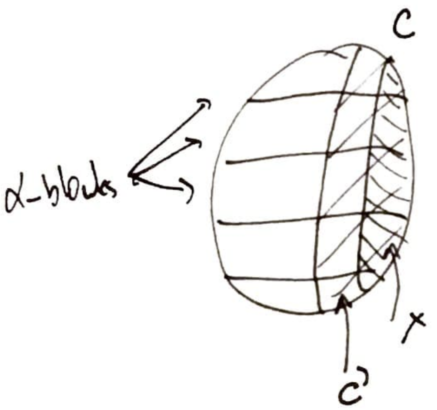
• if $\mathcal{K}$ does not contain but contains nontriv then $\mathcal{K}$ = distrib.

UA $\boxed{7.5}$

THM: $\mathcal{V} = HSP(\underline{A}_1, \dots, \underline{A}_n)$ congruence distributive
$\Rightarrow SI(\mathcal{V}) \subseteq HS(\underline{A}_1, \dots, \underline{A}_n)$

Proof: we only prove that every finite SI $\underline{B}$ is in $HS(\underline{A}_1 \dots \underline{A}_n)$

- take $\underline{B}$ .. a finite SI in $HSP(\underline{A}_1, \dots, \underline{A}_n)$
- $\underline{B} \overset{\text{wloc}}{=} \underline{C}/\alpha \qquad \underline{C} \leq \prod_{i \in I} \underline{D}_i$ each $\underline{D}_i \cong$ some $\underline{A}_j$
- can take C finite:
  - take $X \subseteq_{fin} C$ hitting each α-block
  - $\underline{C}' = Sg_{\underline{C}}(X)$
  - $\underline{C}/\alpha \overset{\text{3rd iso}}{\cong} \underline{C}'/\alpha \restriction C'$
  - $\underline{C}'$ is finite (finitely generated $\Rightarrow$ locally finite)



working in $Con(\underline{C})$

- denote $\pi_i : \underline{C} \longrightarrow \underline{D}_i$ the projection maps
  $\eta_i = ker(\pi_i) \in Con(\underline{C})$
- $\bigwedge_{i \in I} \eta_i = 0_C$ + C finite $\Rightarrow \exists J \subseteq_{fin} I \ \bigwedge_{j \in J} \eta_j = 0_C$
- $\alpha = \alpha \vee 0_C = \alpha \vee \bigwedge_{J} \eta_j \overset{\text{cong. distr. of } \underline{C}!}{=} \bigwedge_{J} (\alpha \vee \eta_j)$ (J finite)
- $\underline{B}$ is SI $\Rightarrow \alpha$ is $\wedge$-irreducible $\Rightarrow \exists j \ \alpha \vee \eta_j = \alpha \Rightarrow \eta_j \leq \alpha$
- $\underline{B} = \underline{C}/\alpha \underset{\eta_j \leq \alpha}{\overset{\text{2nd iso}}{\cong}} \underline{C}/\eta_j \Big/ \alpha/\eta_j \overset{\text{1st iso}}{\cong} \pi_j(\underline{C})/\dots \in HS(\underline{D}_j)$