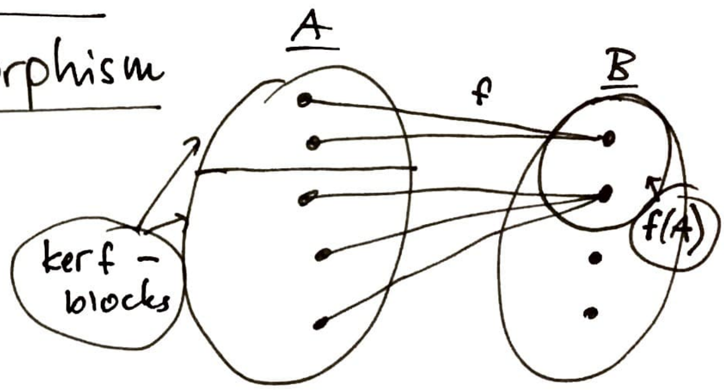
UA 6.1

RECAP

<u>variety</u> = class of algebras (fixed Σ) closed under HSP

$\underline{V(\mathcal{K})}$ = the smallest variety containing $\mathcal{K}$
$= S(\mathcal{K}) \cup HS(\mathcal{K}) \cup \dots \cup PHSPPSP(\mathcal{K}) \cup \dots$
$= HSP(\mathcal{K})$

$f : \underline{A} \longrightarrow \underline{B}$ <u>homomorphism</u>



- $f(A) \le \underline{B}$
- $\ker f \in \mathrm{Con}(\underline{A})$

<u>1st iso Thm:</u>

$$\underline{A}/\ker f \simeq f(\underline{A})$$

TODAY

Decomposing an algebra into simpler ones

- direct decompositions (great, rarely possible)
- subdirect decompositions (good, more often possible)

UA 6.2

DIRECT DECOMPOSITION

Assume $\underline{A} \cong \underline{A}_1 \times \underline{A}_2$

Consider $\pi_1 : \underline{A} \to \underline{A}_1$, $\pi_2 : \underline{A} \to \underline{A}_2$
the natural projection maps

$\eta_1 = \ker \pi_1$ $\qquad$ $\eta_2 = \ker \pi_2$

We have
- $\eta_1, \eta_2 \in \operatorname{Con} \underline{A}$
- $\underline{A}/\eta_1 \cong \underline{A}_1$, $\underline{A}/\eta_2 \cong \underline{A}_2$
- $\eta_1 \wedge \eta_2 = 0_A$
- $\eta_1 \circ \eta_2 = 1_A$ $\quad (\Leftrightarrow \eta_2 \circ \eta_1 = 1_A)$

$\{(a,c) \in A^2 : \exists b \; a \,\eta_1\, b \,\eta_2\, c\}$

Trivial decomposition: $|A_1| = 1$ or $|A_2| = 1$
$(\Leftrightarrow \exists i \; \pi_i \text{ iso} \Leftrightarrow \exists i \; \eta_i = 0 \Leftrightarrow \exists i \; \eta_i = 1)$

THEOREM If $\underline{A}$ algebra, $\alpha, \beta \in \operatorname{Con}(\underline{A})$, $\alpha \wedge \beta = 0$, $\alpha \circ \beta = 1$. Then $\underline{A} \cong \underline{A}/\alpha \times \underline{A}/\beta$ via $a \mapsto (a/\alpha, a/\beta)$.

This decomposition is trivial iff $\alpha = 0$ or $\beta = 0$

Every direct decomposition arises in this way.

UA $\boxed{6.3}$

Def. $\underline{A}$ is directly indecomposable if $\underline{A} \cong \underline{A}_1 \times \underline{A}_2 \Rightarrow \exists i \; |A_i| = 1$

EXAMPLES

- finite directly indecomposable Abelian groups: $\mathbb{Z}_{p^k}$
  — " — Boolean algebras: $\underline{2}$
- directly indecomposable vector spaces over $\underline{F}$: $\underline{F}$
- each finite algebra is isomorphic to a product of directly indecomposable algebras
  e.g. $\mathbb{Z}_{40} \cong \mathbb{Z}_5 \times \mathbb{Z}_8$ (as groups)
- not in general! e.g. $\mathbb{R}^{(\omega)}$
- unique direct decompositions?
  YES : finite groups, rings, lattices
  NO : in general (even finite)

!!!!

WARNING the condition is $\alpha \wedge \beta = 0$, $\alpha \circ \beta = 1$
not $\cancel{\alpha \vee \beta = 1}$

eg. $\underline{A} = (\{1, 2, 3\}$, no op's$)$ or lattice $\underline{A} =$ 

$\alpha = |12|3|$ $\quad \beta = |1|23|$

we have $\alpha \wedge \beta = 0$ $\quad \alpha \vee \beta = 1$ but $\underline{A}$ dir. indecomposable

REMARK finitely generated variety (ie $HSP(\underline{A})$, $\underline{A}$ finite) is directly representable if it has finitely many finite dir. indecomposables

'82 McKenzie – characterization (modulo knowledge of R-Mod, finite R)

UA 6.4

SUBDIRECT DECOMPOSITION

Def. R is a subdirect product of $\underline{A}_i$, $i \in I$ if

$$\underline{R} \leq \prod_{i \in I} \underline{A}_i \text{ and } \forall i \in I \ \pi_i(R) = A_i .$$

Notation: $\underline{R} \leq_{sd} \prod_{i \in I} \underline{A}_i$

$R = \{(1, a, x), (1, b, y), (2, a, z), (2, a, y)\}$

$\pi_1(R) = \{1, 2\}$

$\pi_2(R) = \{a, b\}$

$\pi_3(R) = \{x, y, z\}$

Def. $\underline{R} \leq_{sd} \prod \underline{A}_i$ trivial if

$\exists i \ \pi_i : \underline{R} \to \underline{A}_i$ is an isomorphism

$\Leftrightarrow \pi_i$ injective

$\Leftrightarrow a_i$ determines $(a_1, \dots, a_n) \in R$

Def. $\underline{R}$ is subdirectly irreducible (SI) if it is not isomorphic to a nontrivial subdirect product

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Assume $\underline{R} \leq_{sd} \underline{A}_1 \times \underline{A}_2$

Consider $\pi_1 : \underline{R} \to \underline{A}_1$ $\qquad$ $\pi_2 : \underline{R} \to \underline{A}_2$

$\eta_1 = \ker \pi_1$ $\qquad$ $\eta_2 = \ker \pi_2$

We have

- $\eta_1, \eta_2 \in \operatorname{Con} \underline{R}$
- $\underline{R}/\eta_1 \cong \underline{A}_1$, $\underline{R}/\eta_2 \cong \underline{A}_2$
- $\eta_1 \wedge \eta_2 = 0_R$

Trivial iff $\eta_1 = 0$ or $\eta_2 = 0$

UA $\boxed{6.5}$

THEOREM If $\underline{A}$ algebra, $\alpha_i \in \mathrm{Con}(\underline{A})$, $i \in I$

$\bigwedge_{i \in I} \alpha_i = 0_A$. Then $h : \underline{A} \longrightarrow \prod \underline{A}_i/\alpha_i$ is

$a \longmapsto (a/\alpha_i)_{i \in I}$

an injective homomorphism and

$$\underline{A} \cong h(\underline{A}) \leq_{sd} \prod_{i \in I} \underline{A}_i/\alpha_i$$

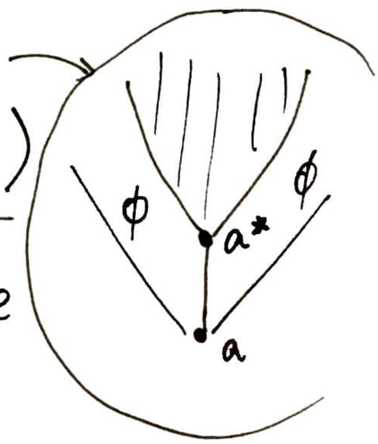
This decomposition is trivial iff $\exists i\ \alpha_i = 0$

Every subdirect decomposition arises in this way

Def. $\underline{L}$ complete lattice. $1 \neq a \in L$ is completely $\wedge$-irreducible

if $\bigwedge_{i \in I} b_i = a \Rightarrow \exists i\ b_i = a$

$\iff \exists a^*$ s.t. $(a < a^*$ and $(\forall b > a)\ b \geq a^*)$



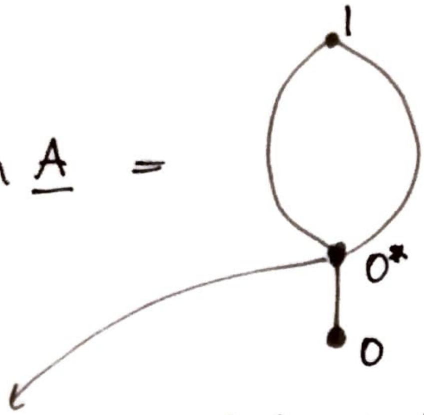
$\underline{A}$ SI iff 0_A is completely $\wedge$-irreducible in Con $\underline{A}$

more generally

$\alpha \in \mathrm{Con}\,\underline{A}$ $\quad \underline{A}/\alpha$ is SI iff α is completely $\wedge$-irreducible in $\mathrm{Con}(\underline{A})$

(UA) $\boxed{6.6}$

$\underline{A}$ SI iff Con $\underline{A}$ =



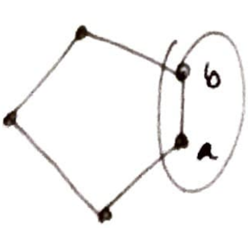
called the **monolith** of $\underline{A}$

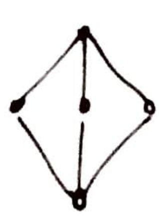
- it is congruence such that $\beta > 0 \Leftrightarrow \beta \geq 0^*$

necessarily $0^* = Cg(a,b)$ for some $a, b \in A$, $a \neq b$

$\underline{A}$ is SI iff $\exists a, b \in A$ $a \neq b$ such that
$\forall \alpha \in \mathrm{Con}\,\underline{A}$ $\alpha \neq 0 \Rightarrow a \overset{\alpha}{\sim} b$

$\boxed{\text{EXAMPLES}}$

- simple (Con $\underline{A} = \{0, 1\}$) $\Rightarrow$ SI $\Rightarrow$ directly indecomposable
- finite Abelian group is SI iff it is $\cong \mathbb{Z}_{p^k}$
- S_n, A_n are SI, $\mathbb{Z}$ is not SI (but 0 is $\wedge$-irreducible -- finitely SI)
- nontrivial distributive lattice is SI iff it is 2-element!
- 

SI  SI (even simple)