(UA) 5.1

RECAP OF BASIC CONSTRUCTIONS

Fix signature Σ

(S) subalgebras $\quad \underline{A} = (A; \dots)$, $B \leq \underline{A}$ non-empty <u>subuniverse</u>

$\rightsquigarrow \underline{B} = (B;$ restrictions $)$

(P) products $\quad \underline{A_i}$, $i \in I$

$\rightsquigarrow \prod \underline{A_i} = (\prod A_i,$ coordinate-wise$)$

(H) quotients $\quad \underline{A}$, $\sim \leq \underline{A}^2$ <u>congruence</u>

$\rightsquigarrow \underline{A}/\sim = (A/\sim;$ using representatives$)$

$Sg_{\underline{A}}(X)$ = subuniverse generated by $X \subseteq A$ $= \bigcap_{X \subseteq B \leq \underline{A}} B$

can be obtained iteratively

$Sg_{\underline{A}}$ .. algebraic closure operator on A

$L_{Sg_{\underline{A}}} =: Sub(\underline{A})$

$Cg_{\underline{A}}(X)$ - congruence generated by $X \subseteq A \times A$ - $\bigcap_{\dots} \dots$

can be obtained iteratively

$Cg_{\underline{A}}$ ... algebraic closure operator on $A \times A$

$L_{Cg_{\underline{A}}} =: Con(\underline{A})$

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VARIETIES

$\mathcal{K}$ ... class of algebras of a fixed signature Σ

$H(\mathcal{K}) :=$ all algebras that are isomorphic to quotients of algebras from $\mathcal{K}$

$S(\mathcal{K}) :=$ .... subalgebras ......

$P(\mathcal{K}) :=$ .... products .... (includes one-element)

$V(\mathcal{K}) := S(\mathcal{K}) \cup H(\mathcal{K}) \cup P(\mathcal{K}) \cup HS(\mathcal{K}) \cup \ldots \cup PHSP(\mathcal{K}) \cup \ldots$

the variety generated by $\mathcal{K}$

= the smallest variety containing $\mathcal{K}$

Def $\mathcal{K}$ is a variety if it is closed under H, S, and P

Examples

✓ • the class of all groups

✓ • —ıı— vector spaces over $\underline{F}$

✓ • —ıı— lattices

all classes "described by identities" are varieties

converse: Birhoff's HSP theorem

× • $\{(Q, \cdot);\ \cdot$ is a group operation$\}$ – not a variety

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- H, S, and P are closure operators
  (on the "class" of all classes of algebras in sign. Σ)
  in particular $SS(\mathcal{K}) = S(\mathcal{K})$, ...

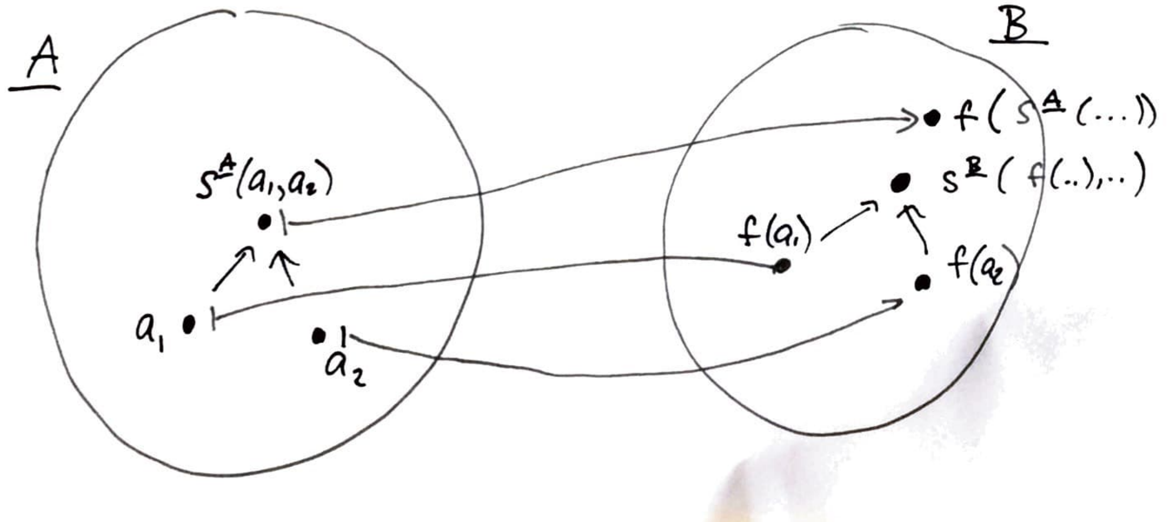
- $SH(\mathcal{K}) \subseteq HS(\mathcal{K})$, $PS(\mathcal{K}) \subseteq SP(\mathcal{K})$
  $PH(\mathcal{K}) \subseteq HP(\mathcal{K})$

- $V(\mathcal{K}) = HSP(\mathcal{K})$!

- $\mathcal{K}$ is a variety iff $HSP(\mathcal{K}) \subseteq \mathcal{K}$

HOMOMORPHISMS

Def. $\underline{A} = (A, \dots)$, $\underline{B} = (B, \dots)$ ... algebras of signature Σ
$f : A \to B$ is a <u>homomorphism</u> from $\underline{A}$ to $\underline{B}$,
written $f : \underline{A} \to \underline{B}$, if it preserves all the basic operations:
$\forall s \in \Sigma \ \forall a_1, \dots, a_{ar(s)} \in A \quad f(s^{\underline{A}}(a_1, \dots)) = s^{\underline{B}}(f(a_1), \dots\dots)$



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Examples

- in groups homo = homo, in vector spaces homo = linear map
- $(S_n, \circ, {}^{-1}, 1) \longrightarrow (\mathbb{Z}_2, +, -, 0)$
  $x \longmapsto \text{parity}(x)$ (0 … even, 1 odd)
- $(\mathbb{R}, +) \longrightarrow (\mathbb{R}, \cdot)$
  $x \longmapsto e^x$
- $(\mathbb{C}, +, \cdot) \longrightarrow (M_2(\mathbb{R}), +, \cdot)$
  $a+bi \longmapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$
- $(\mathbb{Z}, +, \cdot, -, 0, 1) \longrightarrow (\mathbb{Z}_n, +, \cdot, -, 0, 1)$
  $x \longmapsto x \bmod n$

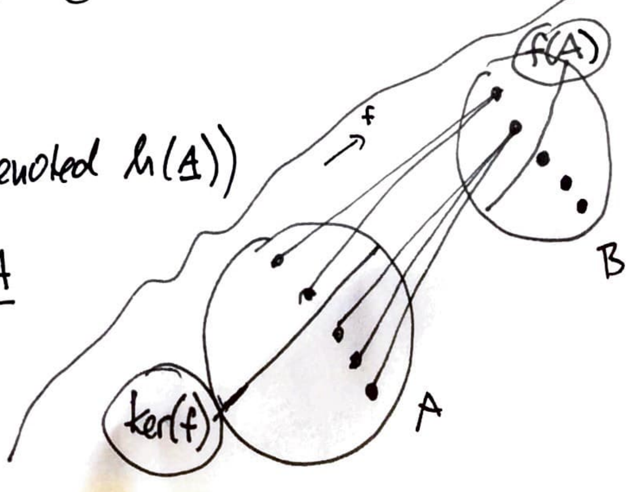
H, S, P & HOMOMORPHISMS

H, S, P ⇝ homo's

$\underline{B} \leq \underline{A}$	$\underline{B} = \underline{A}_1 \times \underline{A}_2$	$\underline{B} = \underline{A}/\sim$
$i : \underline{B} \longrightarrow \underline{A}$ $b \longmapsto b$	$\pi_1 : \underline{B} \longrightarrow \underline{A}$ $(a_1, a_2) \longmapsto a_1$	$q_\sim : \underline{A} \longrightarrow \underline{B}$ $a \longmapsto a/\sim$

Recall: $f : A \to B$ mapping
- $f(A) = \{ f(a) : a \in A \} \subseteq B$
- $\ker(f) = \{ (a, a') : f(a) = f(a') \}$ equiv. on B

(!!!) 0) $h : \underline{A} \to \underline{B}$. Then
- $h(A) \leq \underline{B}$ (the subalg. denoted $h(\underline{A})$)
- $\ker(h)$ is a congruence of $\underline{A}$



Compare to groups

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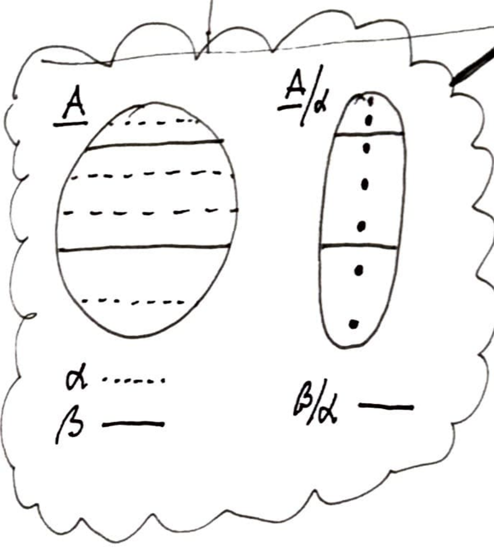
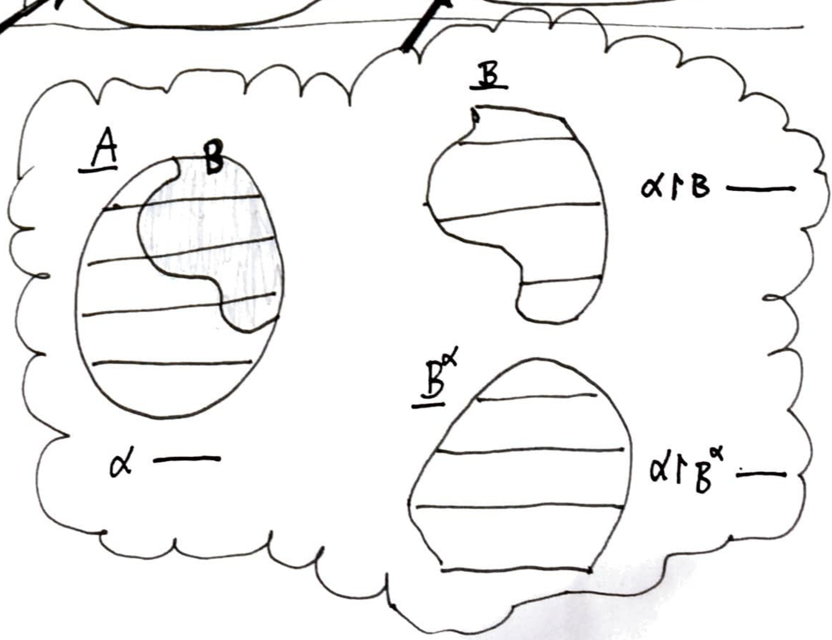
ISOMORPHISM THEOREMS

	1st	2nd (3rd)	3rd (2nd)
groups	$h: \underline{G} \to \underline{H}$ $\Rightarrow h(\underline{G}) \cong \underline{H}/\ker h$	$\underline{K}, \underline{N} \trianglelefteq \underline{G}$ $\underline{K} \leq \underline{N}$ $\Rightarrow \dfrac{\underline{G}/\underline{K}}{\underline{N}/\underline{K}} \cong \underline{G}/\underline{N}$	$\underline{N} \trianglelefteq \underline{G}$ $H \leq \underline{G}$ $\Rightarrow HN/N \cong H/H\cap N$
$n\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$	" $\dfrac{a/c}{b/c} = a/b$ "	$\gcd(a,b)/a = b/\mathrm{lcm}(a,b)$ " ~~$\mathrm{lcm}(a,b)/b = a/\gcd(a,b)$~~ "
general	$h: \underline{A} \to \underline{B}$ $\Rightarrow h(\underline{A}) \cong \underline{A}/\ker h$	$\alpha, \beta \in \mathrm{Con}(\underline{A})$ $\alpha \leq \beta$ $\Rightarrow \dfrac{\underline{A}/\alpha}{\beta/\alpha} \cong \underline{A}/\beta$	$\alpha \in \mathrm{Con}(\underline{A})$ $\underline{B} \leq \underline{A}$ $\Rightarrow \underline{B}^\alpha/\alpha\restriction B^\alpha \cong \underline{B}/\alpha\restriction B$





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ad 2nd $\left(A/\alpha\right)/\left(\beta/\alpha\right) \cong A/\beta$

$\boxed{\text{Correspondence theorem}}$

$\alpha \in \operatorname{Con} \underline{A}$. Then $[\alpha, 1] \longrightarrow \operatorname{Con}(\underline{A}/\alpha)$ is lattice iso

$\beta \longmapsto \beta/\alpha$

interval in $\operatorname{Con} \underline{A}$

"congruences of a quotient are seen in $\operatorname{Con}(\underline{A})$"