UA 3.1.

RECAP

complete lattice, $\exists$ infima $\Rightarrow$ $\exists$ suprema

closure operator ($C: P(X) \to P(X)$, ...)

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closure operator $\longrightarrow$ complete lattice

C $\qquad$ $L_C = (\text{closed}, \subseteq)$

$\bigwedge a = \bigcap a$

$\bigvee a = C(\bigcup a)$

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e.g. $X = G$ for a group $\qquad L_C = (\text{subgroups}, \subseteq)$

$C(A) = Sg(A)$

---

algebraic cl.op. : $\quad C(A) = \bigcup_{B \subseteq_{fin} A} C(B)$

algebraic lat. : $\quad x = \bigvee_{\substack{y \leq x \\ y \text{ compact}}} y$

y compact : $\quad y \leq \bigvee Z \Rightarrow y \leq \bigvee Z'$ for some $Z' \subseteq_{fin} Z$

UA 3.2

$\boxed{\text{THEOREM}}$ C algebraic closure operator on X. Then

(i) L_C algebraic

(ii) compact elements = sets $C(F)$ with F finite

Proof:

- (i) assuming (ii)

$$\underset{L_C}{A} \overset{?}{=} \bigvee_{\substack{B \subseteq A \\ B \text{ compact}}} B$$

$$RHS \overset{\text{Prop}}{=} C\Big(\bigcup_{\substack{B \subseteq A \\ B \text{ compact}}} B\Big) \overset{(ii)}{=} C\Big(\bigcup_{\substack{C(F) \subseteq A \\ F \text{ finite}}} C(F)\Big)$$

$$\overset{A \text{ closed}}{=} C\Big(\bigcup_{F \subseteq_{fin} A} C(F)\Big) \overset{C \text{ alg.}}{=} C(A) = A$$

- F finite $\Rightarrow$ $C(F)$ compact

$$C(F) \subseteq \bigvee \mathcal{a} = C(\bigcup \mathcal{a})$$
$$\overset{?}{\Rightarrow} \exists \mathcal{B} \subseteq_{fin} \mathcal{a} \quad C(F) \subseteq \bigvee \mathcal{B}$$

– $\forall f \in F$ $\exists$ finite $B_f \subseteq \bigcup \mathcal{a}$ $\quad f \in C(B_f)$ $\left(\text{since } C(\bigcup \mathcal{a}) = \bigcup_{B \subseteq_{fin} \bigcup \mathcal{a}} C(B)\right)$

– $B := \bigcup B_f$ we have $B \subseteq_{fin} \bigcup \mathcal{a}$ & $F \subseteq C(B)$

– take $\mathcal{B} \subseteq_{fin} \mathcal{a}$ s.t. $B \subseteq \bigcup \mathcal{B}$

– $C(F) \subseteq C(C(B)) = C(B) \subseteq C(\bigcup \mathcal{B}) = \bigvee \mathcal{B}$

UA 3.3

THEOREM C algebraic closure operator on X. Then
(i) L_C algebraic
(ii) compact elements = sets $C(F)$ with F finite

Proof cntd.

- A compact in $L_C \Rightarrow A = C(F)$ for some finite F

$$A = C(C(A)) \overset{C\text{ alg.}}{=} C\Big(\bigcup_{B \subseteq_{fin} A} C(B)\Big) \overset{\text{Prop}}{=} \bigvee_{B \subseteq_{fin} A} C(B)$$

$$\overset{A\text{ compact}}{\Longrightarrow} \exists B_1, \dots, B_k \subseteq_{fin} A \text{ s.t. } A \subseteq C(B_1) \vee \dots \vee C(B_k)$$

$$F := B_1 \cup \dots \cup B_k$$

$$C(F) = A$$ Exercise

Ex. $\mathrm{Sub}(\underline{G})$ - compact elements = finitely generated subgroups

THEOREM $\forall \underline{M}$ algebraic complete lattice $\exists X$ $\exists C$ algebraic closure operator such that $\underline{M} = L_C$

Proof: $X :=$ compacts of M
$C(A) := \downarrow(\bigvee A) \cap X$

UA

3.4

GALOIS CORRESPONDENCES

- source of closure operators
- give isomorphisms between lattices

Given X, Y sets
$R \subseteq X \times Y$ "relation"

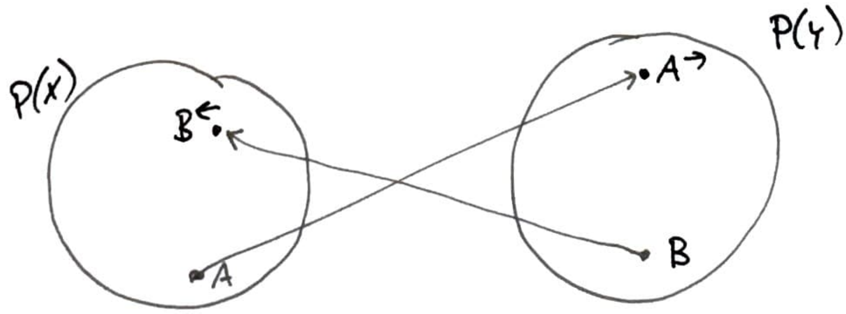
Ex. X people
Y attributes
$(a,b) \in R$ iff a has b

Def. $P(X) \longrightarrow P(Y)$

$$A \longmapsto A^{\rightarrow} := \{ b \in Y ;\ \forall a \in A \ (a,b) \in R \}$$

$P(Y) \longrightarrow P(X)$

$$B \longmapsto B^{\leftarrow} := \{ a \in X ;\ \forall b \in B \ (a,b) \in R \}$$



- $A \subseteq X \Rightarrow A \subseteq A^{\rightarrow\leftarrow}$ $\quad$ $B \subseteq Y \Rightarrow B \subseteq B^{\leftarrow\rightarrow}$
- $A_1 \subseteq A_2 \subseteq X \Rightarrow A_1^{\rightarrow} \supseteq A_2^{\rightarrow}$ $\quad$ ...
- $A \subseteq X \Rightarrow A^{\rightarrow} = A^{\rightarrow\leftarrow\rightarrow}$

Def. the pair $P(X) \underset{\leftarrow}{\overset{\rightarrow}{\rightleftarrows}} P(Y)$ satisfying is a Galois correspondence between X, Y (or PX, PY)

UA $\boxed{3.5}$

$R \subseteq X \times Y$

$A \subseteq X \quad A^{\to} := \{ b \in Y;\ \forall a \in A\ (a,b) \in R \}$

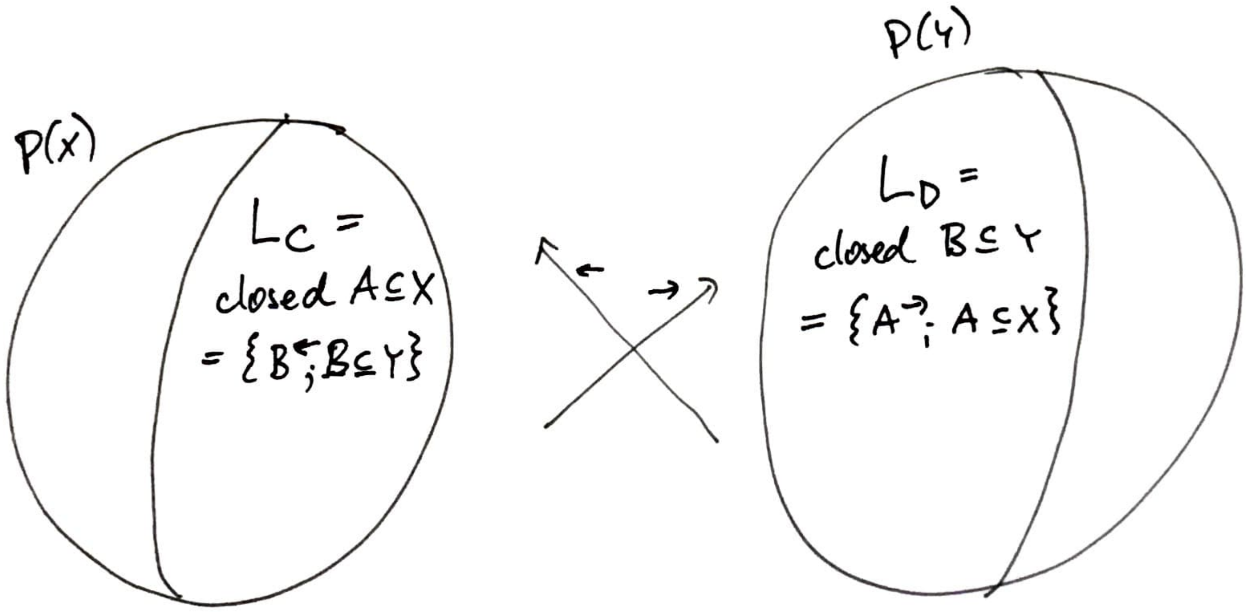
$B \subseteq Y \quad B^{\leftarrow} := \{ a \in X;\ \forall b \in B\ (a,b) \in R \}$

- $A \subseteq A^{\to\leftarrow}$
- $A_1 \subseteq A_2 \Rightarrow A_1^{\to} \supseteq A_2^{\to}$
- $A^{\to} = A^{\to\leftarrow\to}$

<u>Def.</u> $C: P(X) \to P(X)$, $A \mapsto A^{\to\leftarrow}$ $\qquad D: P(Y) \to P(Y)$, $B \mapsto B^{\leftarrow\to}$

- C, D closure operators
- $A \subseteq X$ C-closed iff $A = B^{\leftarrow}$ for some $B \subseteq Y$
- $B \subseteq Y$ D-closed iff $B = A^{\to}$ for some $A \subseteq X$
- $A \mapsto A^{\to}$ is a lattice isomorphism $L_C \to L_D^{\text{dual}}$

  $B^{\leftarrow} \leftarrow\!\shortmid B$ is its inverse



UA 3.6

EXAMPLES OF GALOIS C.

① galois

$X :=$ splitting field of $f \in Q[x]$

$Y := \mathrm{Aut}_{\mathbb{Q}} X$

$(a, f) \in R$ iff $f(a) = a$

closed
- galois subfields of X
- subgroups of Y

} Fundamental theorem of Galois theory

② Hilbert's Nullstellensatz

$X := \mathbb{C}^n$

$Y := \mathbb{Q}[x_1, \dots, x_n]$

$(\bar{a}, f) \in R$ iff $f(\bar{a}) = 0$

closed
- algebraic varieties (by def.)
- radicals of $\mathbb{Q}[x_1, \dots, x_n]$ (Nullstellensatz)

UA $\boxed{3.7}$

3 Mod-Id

$X :=$ algebras in signature Σ (not for purists)
$Y :=$ identities in signature Σ

$(\underline{A}, u \approx v)$ iff $\underline{A}$ satisfies $u \approx v$

closed
- classes closed under H, S, P (Birkhoff's theorem)
- equational theories

4 Pol-Inv

$X :=$ operations on A (finite A)
$Y :=$ relations on A

closed
- clones
- coclones

5 $X = Y =$ vector space of finite dimension with inner product
$(\vec{v}, \vec{w}) \in R$ iff $\vec{v} \perp \vec{w}$

closed = subspaces

6 $X = Y = L$ lattice $(a, b) \in R$ iff $a \leq b$

$L \to L_c$ embedding: Dedekind-MacNeille completion
$x \mapsto \downarrow x$