UA 2.1

I LATTICES, CLOSURE OPERATORS, GALOIS CORRESPONDENCES

many roles of lattices, e.g.

- examples of algebras
- invariants of algebras (e.g. subgroup lattice)
- organizing classes of algebras

lattice $\leftrightarrow$ lattice ordered set

complete lattice $\leftrightarrow$ closure operator $\leftrightarrow$ Galois correspondence
(algebraic) (algebraic)

looser $\leftrightarrow$ than lattice $\leftrightarrow$ lat. o. sets

UA 2.2

LATTICE	$\leftrightarrow$ LATTICE ORDERED SET (a.k.a lattice)
algebra $(A; \wedge, \vee)$ of type $(2,2)$ $\wedge$: meet, $\vee$: join that satisfies ⋮	$(A; \leq)$ where $\leq$ is a p.o. (partial order) s.t. $\forall a,b$ $\sup\{a,b\}$ exists and $\inf\{a,b\}$ exists
$\longrightarrow$	$x \leq y \overset{def}{\equiv} x \wedge y = x$ $\Leftrightarrow x \vee y = y$ then $\sup\{x,y\} = x \vee y$ $\inf\{x,y\} = x \wedge y$
$x \wedge y := \inf\{x,y\}$ $x \vee y := \sup\{x,y$	$\longleftarrow$

☺ $\sup X$, $\inf X$ exist for every <u>nonempty</u> <u>finite</u> $X \subseteq A$

(?) what is $\sup \emptyset$, $\inf \emptyset$?

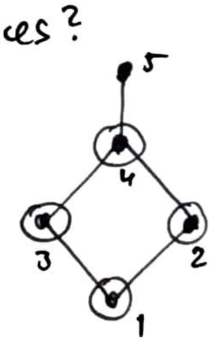
$\boxed{\text{Def}}$ $\mathbb{B} = (B; \wedge^{\mathbb{B}}, \vee^{\mathbb{B}})$ is a <u>sublattice</u> of $\mathbb{A} = (A; \wedge^{\mathbb{A}}, \vee^{\mathbb{A}})$
if $B \subseteq A$ and $\wedge^{\mathbb{B}}$ ($\vee^{\mathbb{B}}$, resp.) is the restriction of $\wedge^{\mathbb{A}}$ ($\vee^{\mathbb{A}}$, resp.) to $B \times B$

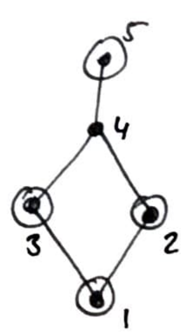
☺ enough to specify B closed under $\wedge, \vee$

(?) sublattices?

$\mathbb{A} = (\{1, \dots, 5\}; \wedge, \vee)$

B ... ☺





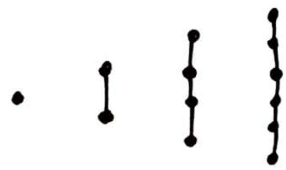
$\nearrow$ sublattice $\neq$ subposet which is a lattice

UA 2.3

EXAMPLES OF LATTICES

often specify $\leq$, what is $\overset{\vee}{\sup}$, $\overset{\wedge}{\inf}$?

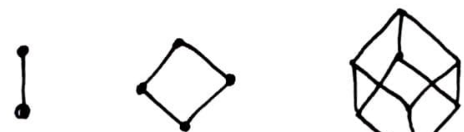
- chains (linear orders) ... $\forall x, y$ $x \leq y$ or $y \leq x$



... $(\mathbb{N}, \leq)$, $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$

- $(P(X); \subseteq)$

a set



$|X|=1$ $|X|=2$ $|X|=3$

- $(Eq(X); \subseteq)$ $Eq(X)$ .. the set of all equivalence relations on X

- $(\mathbb{N}; |)$ $x \mid y$ if y is divisible by x

generalize

- $\underline{G}$ group $(Sub(\underline{G}); \subseteq)$? sublattice of $(P(G), \subseteq)$?

subgroups

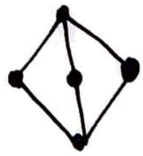
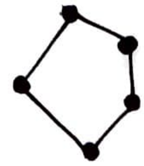
- $\underline{G}$ group $(Con(\underline{G}); \subseteq) \cong (Normal\,Sub(\underline{G}), \subseteq)$

"congruences"

- $\underline{X}$ topological space (e.g. $\mathbb{R}^2$) (open subsets; $\subseteq$)
(closed subsets; $\subseteq$)

? sublattices of $(P(X), \subseteq)$?

UA $\boxed{2.4}$

important examples  

<u>non</u>-examples  

<u>bounded lattice</u>: $\exists\, 0 = \sup \emptyset$
$1 = \inf \emptyset$

usually $(L; \wedge, \vee, 0, 1)$

finite lattice is bounded

<u>dual lattice</u> to $\underset{\underset{\underline{L}}{\shortparallel}}{(L; \wedge, \vee)}$ is $\underset{\underset{\underline{L}^{dual}}{\shortparallel}}{(L; \vee, \wedge)}$

$x \leq y$ in $\underline{L}$ iff $x \geq y$ in $\underline{L}^{dual}$

UA $\boxed{2.5}$

Def Poset $(X, \leq)$ is a <u>complete lattice</u> if
$\forall A \subseteq X$ $\sup A$ and $\inf A$ exist (notation $\bigwedge A := \inf A$, $\bigvee A := \sup A$)

(NON-) EXAMPLES OF COMPLETE LATTICES

× $(\mathbb{Z}; \leq)$ ✓ $(\mathbb{Z} \cup \{\pm\infty\}; \leq)$ ☺ complete $\Rightarrow$ bounded

× $(\mathbb{Q}; \leq)$ × $(\mathbb{Q} \cup \{\pm\infty\}, \leq)$

× $(\mathbb{R}; \leq)$ ✓ $(\mathbb{R} \cup \{\pm\infty\}, \leq)$

✓ finite

× $(\mathbb{N}; |)$ ✓ $(\mathbb{N}_0; |)$

what is it?

<u>complete sublattice</u> $\neq$ sublattice which is complete

✓ $(P(X); \subseteq)$ × $(P_{fin}(X); \subseteq)$ ✓ $(P_{fin}(X) \cup \{X\}, \subseteq)$

finite subsets

✓ $(Eq(X), \subseteq)$ ✓ $Sub(\mathbf{G})$ ✓ $Con(\mathbf{G}) = NormalSub(\mathbf{G})$

$\underline{X}$ topological space ✓ $(closed(\underline{X}); \subseteq)$
✓ $(open(\underline{X}); \subseteq)$

Proposition $(X, \leq)$ poset. If $\forall A \subseteq X$ $\inf A$ exists then $(X; \leq)$ is a complete lattice

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"Proof" $\sup A = \inf\{\text{all upper bounds}\}$
$(= \inf\{x \in X;\ \forall a \in A\ x \geq a\}$

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Note: in some examples • sup works as a "closure"
• elements = closed sets

Def Closure operator on X (a set) is a mapping
$C: P(X) \to P(X)$ such that $\forall A, B \subseteq X$

(i) $A \subseteq C(A)$ "closure of A"

(ii) $C(C(A)) = C(A)$

(iii) $A \subseteq B \Rightarrow C(A) \subseteq C(B)$

Def $A \subseteq X$ is C-closed if $C(A) = A$

EXAMPLES

- $\underline{X} = (X; \dots)$ group $\quad C(A) :=$ subgroup generated by A (denoted $Sg(A)$)
- $C(A) :=$ normal subgroup —"—
- $\underline{X}$ topological space $\quad C(A) = \overline{A}$
- $X = \mathbb{Z} \times \mathbb{Z}$ $\quad C(A) =$ the smallest equivalence relation containg A
- $X =$ sentences of a logic $\quad C(A) =$ all consequences

UA 2.7

(i) $A \subseteq C(A)$
(ii) $C(C(A)) = C(A)$
(iii) $A \subseteq B \Rightarrow C(A) \subseteq C(B)$

Proposition $C: P(X) \to P(X)$ closure operator

(a) $C(A)$ is closed ($\forall A \subseteq X$)
it is the smallest (w.r.t. $\subseteq$) closed set containing A

(b) $\bigcap$ closed sets is closed

(c) ! "closure op. → complete lat."

$L_C :=$ (closed sets, $\subseteq$) is a complete lattice
with $\bigwedge a = \bigcap a$
$\bigvee a = C(\bigcup a)$

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revisit examples

Theorem "complete lattice → closure operator"

$\forall \underline{M}$ complete lattice $\exists X$ $\exists C$ closure operator on X
such that $\underline{M} = L_C$

Proof: $X := M$
$C(A) := \downarrow(\bigvee A)$

iso $f: M \to L_C$
$x \mapsto \downarrow x$

useful: $\underline{K}, \underline{L}$ lattice
$f: K \to L$ bijection s.t.
$k_1 \leq k_2$ iff $f(k_1) \leq f(k_2)$
$\Rightarrow f$ lattice isomorphism

? Is every lattice of the form L_C for some closure operator C?

UA $\boxed{2.8}$

consider $\underline{G}$ group $C(A) = Sg(A)$

i.e. $L_C = Sub(\underline{G})$

(?) Is every complete lattice of the form $Sub(\underline{G})$

ANSWER: NO, e.g. $Sub(\underline{G}) \not\cong (\mathbb{R} \cup \{\pm\infty\}; \ldots)$

these "algebraic" closure operators are special
— they are algebraic

$\boxed{\text{Def.}}$ closure operator C on X is algebraic if

$$\forall A \subseteq X \quad C(A) = \bigcup_{B \subseteq_{fin} A} C(B)$$

Examples: ✓ Sg on a group
× topological examples rarely

$\boxed{\text{Def.}}$ complete lattice $\underline{L}$ is algebraic if

$\forall x \in L$ is a join of compact elements
($\iff$ join of all compact elements below x)

$a \in L$ is compact if $\forall A \subseteq L \;\; a \leq \bigvee A \Rightarrow \exists B \subseteq_{fin} A \;\; a \leq \bigvee B$
"finite-like"

Examples
✓ $(\mathcal{P}(X), \subseteq)$
× $(\mathbb{R} \cup \pm\infty, \leq)$
✓ $Sub(\underline{G})$

(?) compact elements in these examples