UA 1.1

web

subject

classic • general algebraic structures & their classes

"model theory without relations"

modern • clones : sets of operations closed under composition

– generalization of permutation groups to multivariable functions

$\longleftrightarrow$ coclones : sets of relations closed under ...

– link to "model theory without operations"

UA 1.2

why "universal" ?

- covers almost all algebraic structures
  (groups, rings, vector spaces, modules, semigroups, lattices, Boolean algebras, ...)
- generalizes concepts & theorems from special algebras to general algebras
  (subalgebras, products, quotients, homomorphisms, free algebras abelianess, solvability
  isomorphism theorems, Chinese remainder theorem)
  - don't expect miracles for special (classic) kinds of algebras
  - but it's not shallow at all

UA vs. category theory

cats even more general

UA 1.3

short history

- 1930s – 40s

  repeated constructions (eg. free algebras) observed by Birkhoff, Ore, Tarski

  main emphasis : BAs + generalizations (algebraic logic)

- later

  Mal'cev, ... classification of algebras

  Smith, ... abelianess, solvability

  McKenzie, ... structure theory of finite algebras

- 2000– boom coming from a connection to computational complexity (Constrain Satisfaction Problems)

UA 1.4

outline of this course

- lattices, closure operators, galois correspondences
  - useful in general (not just UA)

semantics
- basic constructions (H, S, P, iso thms.)
- direct & subdirect decomposition

syntex
- free algebras, Mod - Id galois correspondence, Birkhoff's HSP theorem

modern
- clones & coclones

classic
- Mal'cev conditions

UA $\boxed{1.5}_2$

Notation

$$A^B = \{f;\ f: B \to A\}$$
$$A^n = A^{\{1,2,\dots,n\}} = \text{n- tuples of elements of } A$$
$$A^0 = A^\emptyset = \{\emptyset\}$$

<u>n-ary operation on A</u> : $f: A^n \to A$

<u>n-ary relation on A</u> : $R \subseteq A^n$

- binary
- unary
- nullary
  = constant
- ternary

2-3 common formalisms

(1) <u>type</u> .. a tuple $(ar_1, ar_2, \dots, ar_k)$

<u>algebra of type</u> $\underline{A} = (A;\ f_1, \dots, f_k)$

<u>universe</u>
<u>domain</u> ...

nonempty!

operation on A of arity ar_1

(2) <u>signature</u> $\begin{cases} \Sigma & \text{set of symbols} \\ ar: \Sigma \to \mathbb{N}_0 \end{cases}$

<u>algebra of signature Σ</u> $\underline{A} = (A;\ (f^{\underline{A}})_{f \in \Sigma})$

$f^{\underline{A}}$ operation of arity $ar(f)$

(3) signature-free $(A,$ set of operations$)$

UA $\boxed{1.6}$

Examples

classic obvious

- group type $(2,1,0)$ sign. $\Sigma = \{\cdot, ^{-1}, 1\}$
- ring
- semigroup
- × fields

classic not so obvious

- vector spaces

  but $\cdot : F \times V \to V$ not operation

  $\to \forall f \in F \quad \cdot_f : V \to V$ is unary op.

  $v \mapsto fv$

- R-modules

combinatorial

- quasigroups $\leftrightarrow$ latin square

  latin square: $(a_{ij})_{i,j \in I}$ a_{ij}

  each row has each element exactly once

  each column — " —

  $i * j := a_{ij}$

  $\forall i,k \ \exists! j \ a_{ij} = k$ i.e. $i * j = k$ denote $j := i \backslash k$

  $\forall j,k \ \exists! i \ a_{ij} = k$ $\quad i := k/j$

  latin squares $\leftrightarrow$ algebras $(I; *, \backslash, /)$ + axioms $\leftarrow$ quasigroup

- grupoids of other kinds

UA | 1.7

ordered

- semilattice $\longleftrightarrow$ semilattice-ordered sets
  $(S; \wedge)$ $\quad$ $(S; \leq)$
  partial order where $\inf\{x,y\}$ exists
- lattice $\longleftrightarrow$ lattice-ordered sets
  $(L; \wedge, \vee)$ $\quad$ $(L; \leq)$
  $\sup\{x,y\}$, $\inf\{x,y\}$ exist
- ℓ-group $(G; \cdot, ^{-1}, 1, \wedge, \vee)$

× ordered groups

- Boolean algebras $(B; \wedge, \vee, \neg, 0, 1)$

signature-free

- G-sets $(X;$ set of (unary) bijections$)$
- vector spaces $(X;$ all linear forms$)$

"The same" algebras

many meanings

① term equivalence

$\underline{A} = (A; \ldots) \overset{\text{term}}{\sim} \underline{B} = (A; \ldots)$

if they have the same term operations
– operations obtained by composing

② polynomial equivalence

ditto, + we can use elements from A

Ex.
- $(G; \cdot, ^{-1}, 1)$ as a group $\overset{\text{(almost) term}}{\sim}$ $(G; \cdot, /, \backslash)$ as a quasigroup
- $(V; +, -, 0, a\cdot) \overset{\text{term}}{\sim} (V;$ all linear forms$)$
- $(\{0,1\}; \wedge, \vee, \neg) \overset{\text{term}}{\sim} (\{0,1\}; \mid)$ nand $\overset{\text{term}}{\sim} (\{0,1\};$ all $)$

(UA) $\boxed{1.9}$

"the same" cntd.

③ <u>isomorphism</u>

$\underline{A}$, $\underline{B}$ the same signature

$\underline{A} = (A; \dots) \simeq \underline{B} = (B; \dots)$ if

isomorphic

$\exists \varphi : A \to B$ bijection

$\forall f \in \Sigma \quad \varphi f^{\underline{A}}(a_1, \dots, a_n) = f^{\underline{B}}(\varphi a_1, \dots, \varphi a_n)$

- $\underline{B}$ is obtained from $\underline{A}$ by renaming elements

<u>Examples</u>

- $(\{0,1\}, +_{\text{mod } 2}) \simeq (\{1,-1\}; \cdot)$
- $(\mathbb{N}; +) \not\simeq (\mathbb{R}; +)$
- $(\mathbb{N}; +) \not\simeq (\mathbb{Q}; +)$