UA $\boxed{12.1}$

RECAP

Absorption theorem: $\underline{A}, \underline{B}$ finite Taylor
$R \leq_{sd} \underline{A} \times \underline{B}$ linked
Then $\exists C \trianglelefteq \underline{A}$ or $\exists D \trianglelefteq \underline{B}$

$\boxed{\text{how to apply it:}}$
"no absorption $\Rightarrow$ no linked relation"

Ex: finite, abelian, Taylor $\Rightarrow$ affine
Proof: abelian $\Rightarrow$ HAF $\Rightarrow$ Malcev $\Rightarrow$ affine

TODAY

" linked relation $\Rightarrow$ absorption"

Theorem: $\underline{A}$ finite Taylor, $R \leq_{sd} \underline{A} \times \underline{A}$ linked.
Then $\exists a \in A \ (a,a) \in R$. $\boxed{\text{A loop lemma}}$

Consequences:
- dichotomy for CSP(graph)
- $\underline{A}$ finite idempotent. (a)
  (i) $\underline{A}$ Taylor
  (ii) $\exists s \in \mathrm{Clo}(\underline{A}) \ s(xyxzyz) \approx s(yxzxzy)$

UA 12.2

Absorbing linkedness

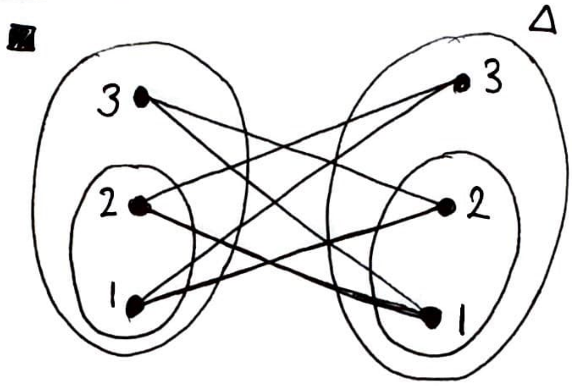
Example

- $\underline{A} = (A; \ldots)$ ⟶ idempotent
- $R \leq \underline{A} \times \underline{A}$

$A = \{1,2,3\}$

$R = A^2 \setminus \{(1,1),(2,2),(3,3)\}$
$(= \neq)$

What if $\{1,2\} \trianglelefteq \underline{A}$? (Say by t)



■ $t(1111 \ldots 1) = 1 \; a_0$

△ $t(3222 \ldots 2) = b_1 \in \{1,2\} \trianglelefteq \underline{A}$

■ $t(2111 \ldots 1) = a_1 \in \{1,2\} \leq \underline{A}$

△ $t(1322 \ldots 2) = b_2 \in \{1,2\} \trianglelefteq \underline{A}$

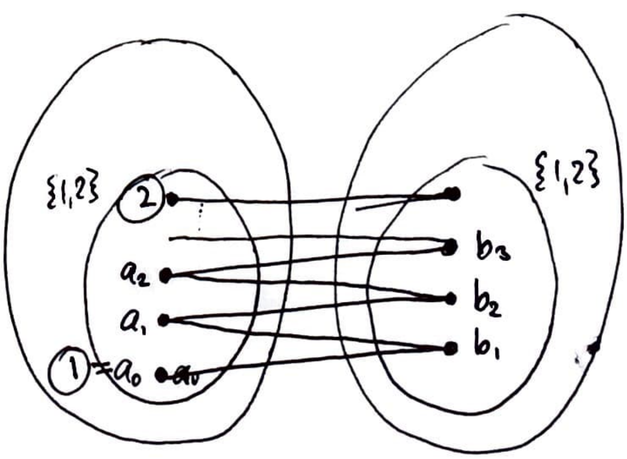
■ $t(2211 \ldots 1) = a_2$

△ $t(1132 \ldots 2) = b_3$

■ $t(2221 \ldots 1) = a_3$

⋮

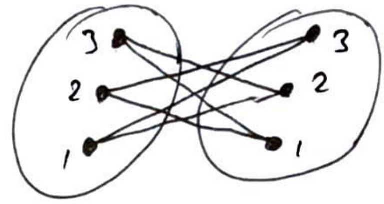
■ $t(222 \ldots 2) = 2 \; a_{\ldots}$



1, 2 on the left are linked ... ↯

(in fact $a_i = 1$ $b_i = 2$ but we don't need it)

$\boxed{\text{UA}}$ $\boxed{12.3}$



Walking

Example

- $\underline{A} = (\{1,2,3\}, \cdots)$
- $R \leq \underline{A} \times \underline{A}$ $\quad R = A^2 \setminus \{(1,1),(2,2),(3,3)\}$

What if $\{1\} \trianglelefteq \underline{A}$?

Then $\{1\}+R := \{ b \in A;\ (1,b) \in R \} \trianglelefteq \underline{A}$

$$t(\overset{\in}{b_1}, \overset{\in}{b_2}, \dots, \underset{A}{\overset{c}{\cap}}, b_{\dots}, \dots) \overset{?}{\in} \{1\}+R$$

$$\begin{aligned} t(1,\ 1,\ \dots,\ d,\ 1,\ \dots 1) &= 1. \quad (\text{as } \{1\} \trianglelefteq_t \underline{A}) \\ t(b_1,\ b_2,\ \dots,\ c,\ b_{\dots},\ \dots) &= \bullet \in \{1\}+R \end{aligned}$$

in this example $\{1\}+R = \{2,3\} \trianglelefteq \underline{A}$ ↯

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in general $\quad R \leq \underline{A} \times \underline{B}$

$C \trianglelefteq \underline{A} \Rightarrow C+R := \{ d \in B;\ \exists c \in C\ (c,d) \in R \}$
$\trianglelefteq \underline{B}$

$D \trianglelefteq \underline{B} \Rightarrow D-R := \{ c \in A;\ \exists d \in D\ (c,d) \in R \}$
$\trianglelefteq \underline{A}$

$$\begin{aligned} t(c_1,\ c_2 \quad ,\bigcirc\ c_{\dots} \quad) &\in C \\ &\Downarrow \\ t(d_1,\ d_2,\ \dots,\bigcirc,\ d_{\dots} \quad) &\in C+R \end{aligned}$$

UA 12.4

Subdirect part

$R \subseteq A^2$. <u>Subdirect part</u> of R is

$$\{ a_0 \in A ;\ \exists a_{-2}, a_{-1}, a_1, a_2, \dots \quad (a_{-2}, a_{-1}), (a_{-1}, a_0), (a_0, a_1), (a_1, a_2) \dots \in R \}$$

$$\cdots \ \underset{a_{-2}}{\bullet} \longrightarrow \underset{a_{-1}}{\bullet} \longrightarrow \underset{a_0}{\bullet} \longrightarrow \underset{a_1}{\bullet} \longrightarrow \underset{a_2}{\circ} \ \cdots$$

now we draw R as a digraph (not a bipartite graph)

It is the largest $C \subseteq A$ such that

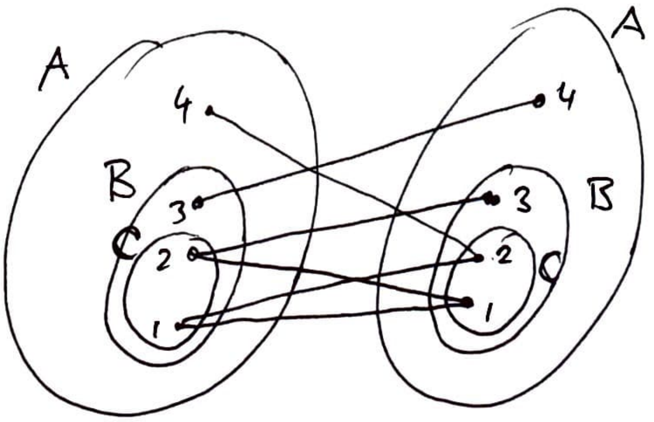
$$R \cap (C \times C) \subseteq_{sd} C \times C$$

If A finite, then the subdirect part of R is

$\{ a_0 \in A ;\ a_{-n}, \dots\dots, a_n \dots \}$ for $\forall$ sufficiently large n

If $R \leq_{sd} \underline{A}^2$, $\underline{B} \trianglelefteq \underline{A}$, $S = R \cap (B \times B)$

Then (subdirect part of S) $\trianglelefteq \underline{A}$



Exercises

UA 12.5

A loop lemma

Theorem $\underline{A}$ finite Taylor, $R \leq_{sd} \underline{A}^2$ linked.
Then $\exists a \in A \; (a,a) \in R$

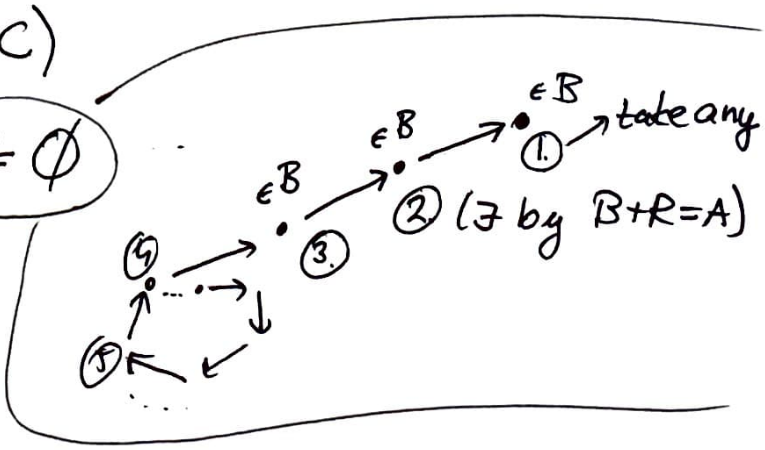
[Barto, Kozik, Niven '00s]

Proof:
- Take a counterexample with minimal $|A|$
- $\exists \underline{D} \ntrianglelefteq \underline{A}$ (absorption theorem)
- $D + R - R + R - R + \cdots\cdots = A$ for a sufficient length
- $D + R \trianglelefteq \underline{A}$, $D + R - R \trianglelefteq \underline{A}$, ...
- $\exists \underline{B} \ntrianglelefteq \underline{A}$ such that $B + R = A$ or $B - R = A$

WLOG

- $S := R \cap (B \times B)$, $C :=$ subdirect part of S

$T := S \cap (C \times C)$
- $C \ntrianglelefteq \underline{A}$, $C \neq \emptyset$



- $T \leq_{sd} \underline{C} \times \underline{C}$ is linked

(absorbing linkedness – Exercise) $\lightning$

UA 12.6

Dichotomy for CSPs over graphs

Theorem Let $\mathbb{R} = \{R\}$, where $R \subseteq A \times A$ symmetric, A finite.

If R contains a loop ($(a,a) \in R$) or is bipartite, then $CSP(\mathbb{R})$ in P

Else $CSP(\mathbb{R})$ is NP-complete. [Hell, Nešetřil '90s]

Proof: • $\mathbb{A} := (A; R)$ recall $CSP(\mathbb{R}) \sim Hom(\mathbb{A})$

• $\mathbb{B} := (B; S)$ the core of $\mathbb{A}$, i.e.

  • $\exists$ homo $\mathbb{A} \to \mathbb{B}$, $\exists$ homo: $\mathbb{B} \to \mathbb{A}$

  • every endomorphisms of $\mathbb{B}$ is bijective

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Note: $Hom(\mathbb{A}) = Hom(\mathbb{B})$

• if $(a,a) \in R$, then $\mathbb{B} =$ 

if R bipartite, then $\mathbb{B} =$ 

both cases $Hom(\mathbb{B}) = Hom(\mathbb{A})$ in P

• else it contains an odd cycle (& no loop)

  • for simplicity: assume S connected

  • $S \leq_{sd} B \times B$ linked (Exercise)

  • $\mathcal{B} := Pol_{idempotent}(\mathbb{B})$, $\underline{B}$ s.t. $\mathcal{B} = Clo(\underline{B})$

  • $S \leq_{sd} \underline{B} \times \underline{B}$ linked $\overset{\text{loop lemma}}{\Longrightarrow}$ $\underline{B}$ not Taylor

  • $Pol(\mathbb{B}) \overset{min}{\sim} Pol_{idempotent}(\mathbb{B}) \overset{min}{\sim} Proj$

$\Longrightarrow Hom(\mathbb{B}) = Hom(\mathbb{A})$ NP-complete

UA 12.7

Sigger's operation

Theorem Let $\underline{A}$ be finite idempotent. TFAE [Siggers '10s]

(i) $\underline{A}$ is Taylor

(ii) $\exists s \in Clo_6(\underline{A})$ $\; s(x\,y\,x\,z\,y\,z) \approx s(y\,x\,z\,x\,z\,y)$

(ii) $\Rightarrow$ (i) $\quad s$ is a Taylor operation

(i) $\Rightarrow$ (ii)

- $\underline{F} = Clo_3(\underline{A}) \leq \underline{A}^{A^3}$
  - $x := \pi_1^3,\; y := \pi_2^3,\; z := \pi_3^3$
  - recall $\underline{F} = Sg_{\underline{A}^{A^3}}(\{x,y,z\}) \cong F_{\underline{A}}(\{x,y,z\})$
- $\underline{F}$ Taylor
- take $R = Sg_{\underline{F}^2}\left(\left\{ \binom{x}{y}, \binom{y}{x}, \binom{x}{z}, \binom{z}{x}, \binom{y}{z}, \binom{z}{y} \right\}\right) \leq \underline{F} \times \underline{F}$
  - $R \leq_{sd} F \times F$ (since x,y,z generate $\underline{F}$)
  - R is linked (since generators are )
- Loop lemma $\Rightarrow$ R has a loop ($(a,a) \in R$ for some a)

$\Rightarrow \exists s \quad s^{\underline{F}^2}\left(\binom{x}{y}, \binom{y}{x}, \binom{x}{z}, \binom{z}{x}, \binom{y}{z}, \binom{z}{y}\right) = \binom{a}{a}$

$\Rightarrow s^{\underline{A}}(x\,y\,x\,z\,y\,z) = s^{\underline{A}}(y\,x\,z\,x\,z\,y)$