UA $\boxed{11.1}$

# RECAP

<u>Taylor algebra $\underline{A}$</u> : $\underline{A}$ idempotent & $Clo(\underline{A}) \overset{(min)}{\not\to} Proj$

<u>B absorbs $\underline{A}$</u> (idempotent): $B \leq \underline{A}$ & $\exists t \in Clo(\underline{A})$

$B \trianglelefteq_t \underline{A}$ $\qquad \forall i$ $t(B, B, \dots, B, \underset{i}{A}, B, B, \dots, B) \subseteq B$

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<u>Absorption theorem</u> : $\underline{A}, \underline{B}$ finite Taylor (same $\Sigma$)

$R \lneq_{sd} \underline{A} \times \underline{B}$ linked

Then $\exists \underline{C} \ntrianglelefteq \underline{A}$ or $\exists \underline{D} \ntrianglelefteq \underline{B}$

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Proof: • WLOG $R$ left central

$\underline{B}$ — projective $\overset{+\text{Taylor}}{\Longrightarrow}$ 2-absorption

$\underline{B}$ — transitive $t^{\underline{B}}$ $\Rightarrow$ left center $\trianglelefteq_{t^{\underline{A}}} \underline{A}$

# TODAY

Finite abelian Taylor $\Rightarrow$ affine

Bonus material: cube operations

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## HAF stable under $HSP_{fin}$

**Def.** Idempotent $\underline{A}$ is HAF (hereditarily absorption free) if $\underline{C} \trianglelefteq \underline{B} \leq \underline{A} \Rightarrow C = B$

ie. no non-trivial absorption in any subalgebra

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**Proposition** The class of idempotent HAF algebras (of a fixed signature) is closed under $H, S, P_{fin}$

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Proof (S) ✓

(H) $C \trianglelefteq_{t^{\underline{A}/\alpha}} \underline{A}/\alpha \Rightarrow \bigcup C \trianglelefteq_{t^{\underline{A}}} \underline{A}$

(P) Say $\underline{C} \trianglelefteq \underline{B} \trianglelefteq \underline{A} \times \underline{A}'$ ; $\underline{A}, \underline{A}'$ HAF

- $\text{proj}_1(C) \trianglelefteq \text{proj}_1(\underline{B}) \leq \underline{A}$ (Exercise)
  $\Rightarrow \text{proj}_1(C) = \text{proj}_1(B)$
- take $(a, a') \in B$
- consider $D = \{a'' ; (a, a'') \in B\}$ $(\ni a')$
  $E = \{a'' ; (a, a'') \in C\}$
- $E \neq \emptyset$, $E \trianglelefteq \underline{D} \leq \underline{A}' \Rightarrow E = D \Rightarrow (a, a') \in C$

(Exercise)

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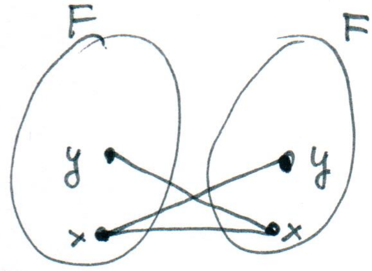
# $\boxed{\text{Taylor + HAF} \Rightarrow \text{Mal'cev}}$

**Proposition** If $\underline{A}$ is finite, Taylor, and HAF, then $\underline{A}$ has a Mal'cev term operation

Proof

- Take $\underline{F} = \underline{Clo}_2(\underline{A}) \leq \underline{A}^{A^2}$ (iso to $F_{\underline{A}}(\{x, y\})$)
  - denote $x = \pi^2_1$, $y = \pi^2_2$
  - recall $F = Sg_{\underline{A}^{A^2}}(\{x, y\})$
- previous proposition $\Rightarrow$ $\underline{F}$ is HAF
- take $R = Sg_{\underline{F}^2}\left(\left\{\binom{y}{x}, \binom{x}{x}, \binom{x}{y}\right\}\right) \leq \underline{F} \times \underline{F}$
  - $R \leq_{sd} F \times F$ (since $x, y$ generate $\underline{F}$)
  - $R$ is linked (since generators of $R$ linked in $\{x, y\}$ - exercise)
- Absorption theorem + $\underline{F}$ HAF and Taylor $\Rightarrow$ $R = F \times F$




- $\binom{y}{y} \in R \Rightarrow \exists t \quad t^{\underline{F}^2}\left(\binom{y}{x}, \binom{x}{x}, \binom{x}{y}\right) = \binom{y}{y}$

$\Rightarrow t^{\underline{A}}(y, x, x) = y, \quad t^{\underline{A}}(x, x, y) = y$

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# Abelian $\Rightarrow$ HAF

**Proposition** If $\underline{A}$ is finite idempotent, abelian then $\underline{A}$ is HAF.

Proof
- enough to show $\underline{B} \trianglelefteq \underline{A} \Rightarrow B = A$
  sinse subalgebra of abelian alg. is abelian
- will show $\underline{B} \trianglelefteq_t \underline{A}$ where $t$ $n$-ary, $n \geq 2$
  $\Rightarrow \underline{B} \trianglelefteq_s \underline{A}$ where $s$ $(n-1)$-ary
  enough sinse $\underline{B} \trianglelefteq_s \underline{A}$ $s$ unary $\Rightarrow B = A$
- $n = 2$ (for simplicity)
- define $t_m(x,y) = \underbrace{t(x, t(x, t(\ldots, t(x,y)))\ldots)}_{m}$
  - $B \trianglelefteq_{t_m} \underline{A}$ (thm)
  - for $m = |A|!$ $\quad t_m(x, t_m(x,y)) = t_m(x,y)$
    why? • $\forall x$ $\quad t_m(x,y) = \underbrace{s_x \circ \ldots \circ s_x}_{m}(y)$ $\quad s_x(y) = t(x,y)$
    • $\forall$ unary $s: A \to A$ $\quad \underbrace{s \circ s \ldots \circ s}_{m}(y) = \underbrace{s \circ s \circ \ldots \circ s}_{2m}(y)$
- $t_m(\underline{b}, t_m(b,a)) = t_m(\underline{b}, a)$ $\quad a \in A, b \in B$
  $\Downarrow$ abelianess
  $t_m(a, \underset{\substack{\cap \\ B}}{t_m(b,a)}) = t_m(a,a)$ $\quad$ so $s(x) = t(x,x)$ works

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# Abelian $\Rightarrow$ HAF contd.

- In general, assume

  $\mathbf{B} \trianglelefteq_t \underline{A}$ where $t$ $n$-ary

- define

  $$t_m(x_1, \dots, x_{n-1}, y) = t(x_1, \dots, x_{n-1}, t(x_1, \dots, x_{n-1}, t(\dots \\ \dots, t(x_1, \dots x_{n-1}, y)) \dots)$$

  - $\mathbf{B} \trianglelefteq_{t_m} \underline{A}$
  - $m = |A|!$ $\quad t_m(x_1, \dots, x_{n-1}, t_m(x_1, \dots, x_{n-1}, y)) = t(x_1, \dots, y)$

- $s(x_1, \dots, x_{n-1}) := t_m(x_1, \dots, x_{n-1}, x_{n-1})$
- $s(B, \dots, B, A) \overset{?}{\in} B$ (rest is clear)

  $$t_m(b_1, \dots, b_{n-2}, \underline{b_{n-1}}, t_m(b_1, \dots, b_{n-1}, a)) = t_m(b_1, \dots, b_{n-2}, \underline{b_{n-1}}, a)$$

  $\Downarrow$ abelianess

  $$\underbrace{t_m(b_1, \dots, b_{n-2}, \underline{a}, t_m(b_1, \dots b_{n-1}, a))}_{\in B} = t_m(b_1, \dots, b_{n-2}, \underline{a}, a) = s(b_1, \dots, b_{n-2}, a)$$

  (here $b_1, \dots, b_{n-1} \in B$ $a \in A$ arbitrary)

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$\boxed{\text{Abelian + Taylor} \Rightarrow \text{affine}}$ for finite!

**Theorem** [Hobby, McKenzie '80s
this proof: Barto, Kozik, Stanovský]

If $\underline{A}$ is finite, abelian, and Taylor,
then $\underline{A}$ is affine.

Proof: (i) • $\underline{A}$ is HAF (abelian $\Rightarrow$ HAF)

(ii) • $\underline{A}$ has a Mal'cev term operation
(Taylor + HAF $\Rightarrow$ Mal'cev)

(iii) • $\underline{A}$ is affine (Fundamental theorem:
abelian + Mal'cev $\Rightarrow$ affine)

**Corollary** If $\underline{A}$ is finite and has an idempotent Taylor operation, then $\underline{A}$ is affine

Prf: apply (i), (ii) to $\underline{A}' = (A; \text{Taylor op.})$

not true for infinite: $(\mathbb{Q}; \frac{x+y}{2})$

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# Bonus: cube operations

**Theorem** [part by Marković, Maróti, McKenzie 2010s
Barto, Kozik, Stanovský]

$\underline{A}$ finite idempotent. TFAE

(1) No subalgebra of $\underline{A}$ has a nontrivial projective subuniverse

(2) Every subalgebra of $\underline{A}$ has a transitive term operation

(3) $Clo(\underline{A})$ contains a cube operation $t$, i.e. it satisfies identities of the form

$$t(x, *, \dots *) \approx y$$
$$t(*, x, *, \dots) \approx y$$
$$\vdots$$
$$t(*, \dots *, x) \approx y$$

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Examples: Mal'cev, near unanimity

Proof: (1) $\Leftrightarrow$ (2) ✓

(3) $\Rightarrow$ (1) ✓

(UA) $\boxed{11.8}$

$(1) \Rightarrow (3)$ no subalgebra of $\underline{A}$ has a nontrivial projective subuniverse $(*)$

$\Rightarrow$ $\underline{A}$ has a cube term operation

- $(*)$ is stable under $HSP_{fin}$ (proved liked for HAF)

- $\underline{F} := F_{\underline{A}}(\{x, y\})$

  satisfies $(*)$ $\Rightarrow$ has transitive term operation $s^{\underline{F}}$

$$s(x, f_2^1(x,y), \ldots, f_n^1(x,y)) \approx y$$

$$s(f_1^2(x,y), x, f_3^2(x,y), \ldots) \approx y$$

$$\ldots$$

$$s(f_1^n(x,y), \ldots, f_{n-1}^n(x,y), x) \approx y$$

- $t = s(\, f_1^2 \cancel{(x,y)} * f_1^2 * \ldots * f_1^n,$

  $f_2^1 * f_2^3 * \ldots f_2^n,$

  $\ldots,$

  $f_n^1 * f_n^2 * \ldots * f_n^{n-1})$ is

a cube operation