UA $\boxed{10.1}$

RECAP

$\underline{A}$ Taylor if $Clo(\underline{A})$ is

$\underline{\text{Taylor clone}}$: $\mathcal{A}$ idempotent & $\mathcal{A} \not\to Proj$

$\mathcal{A}$ idempotent ⇕

- $\mathcal{A} \not\to Proj$ ($\Leftrightarrow \mathcal{A}$ Taylor) "non-trivial identities"
- $Proj \in HSP(\mathcal{A})$
  for finite $Proj \in HS(\mathcal{A})$
- $\mathcal{A} \overset{min}{\not\to} Proj$
- $\mathcal{A}$ contains a Taylor operation
  $t(\dots, \underset{i}{x}, \dots) \approx t(\dots, y, \dots)$

even for finite non-idempotent

NOW

Absorbing subalgebras

- subalgebras with additional property
- often exist in Taylor algebras
  (e.g. absorption theorem)
- useful when studying ~~R~~ $Inv(\underline{A})$

UA 10.2

or "$\underline{B}$ absorbs $\underline{A}$ by t" idempotent!

Def. $\underline{B}$ is an absorbing subalgebra of $\underline{A}$ if $\underline{B} \leq \underline{A}$ and $\exists t \in Clo(\underline{A})$ such that

$$t(A, B, B, \dots, B) \subseteq B$$
$$t(B, A, B, \dots, B) \subseteq B$$
$$\dots$$
$$t(B, B, \dots B, A) \subseteq B$$

set theoretical image

(i.e. $t(a_1, \dots, a_n) \in B$ whenever all but at most 1 a_i is in B)

Written $\underline{B} \trianglelefteq_t \underline{A}$ or $\underline{B} \trianglelefteq \underline{A}$.

Note $\supseteq$ automatic from idempotency

Examples

- If $\underline{A}$ has a majority operation or near-unanimity operation ($t(y, x, \dots x) \approx \dots \approx t(x, \dots x, y) \approx x$) then $\forall a \in A$ $\{a\} \trianglelefteq \underline{A}$
- If $\underline{A}$ has a semilattice operation $\wedge$, and is finite then $\{\min\} \trianglelefteq \underline{A}$
- The only absorbing subuniverses of $(\mathbb{Z}_n, x-y+z)$ are trivial ($\emptyset, A$)

Terminology: n-absorbing .... absorbing by n-ary t

UA $\boxed{10.3}$

not necessary

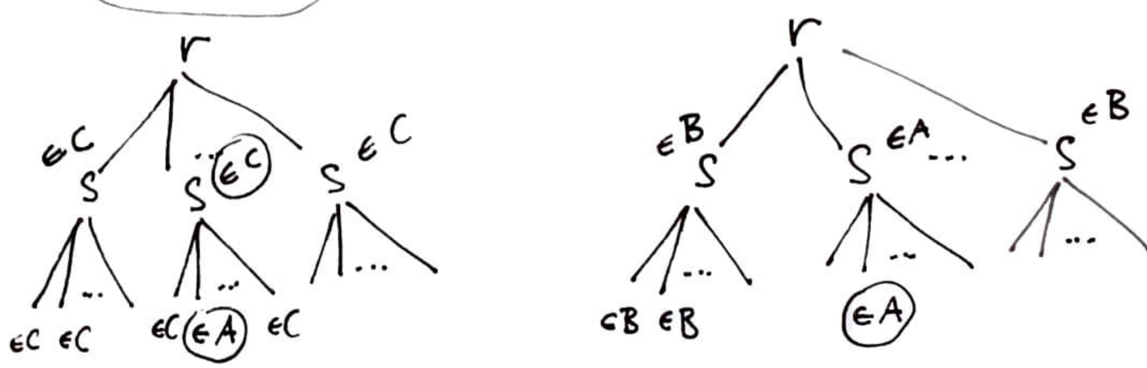
$\boxed{\text{Proposition}}$ (Basic properties) $\underline{A}$ idempotent.

- $B, C \trianglelefteq \underline{A} \Rightarrow \exists t \in Clo(\underline{A})\ B \trianglelefteq_t \underline{A},\ C \trianglelefteq_t \underline{A}$
- $B, C \trianglelefteq \underline{A} \Rightarrow B \cap C \trianglelefteq \underline{A}$
- $\underline{C} \trianglelefteq \underline{B} \trianglelefteq \underline{A} \Rightarrow \underline{C} \trianglelefteq \underline{A}$

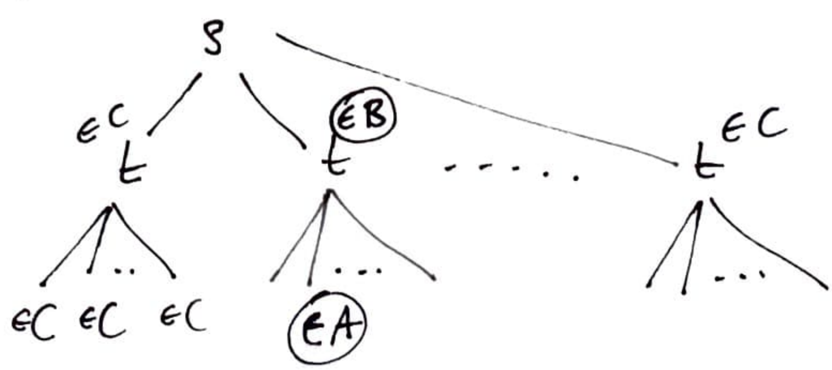
Proof: • common witness

- $B \trianglelefteq_r \underline{A},\ \underline{C} \trianglelefteq_s \underline{A} \Rightarrow B, C \trianglelefteq_t \underline{A}$

$t = r * s$



- intersection: clear for $B, C \trianglelefteq_t \underline{A}$ use ↑
- transitivity: $\underline{C} \trianglelefteq_s \underline{B} \trianglelefteq_t \underline{A} \Rightarrow \underline{C} \trianglelefteq_{s*t} \underline{A}$



UA 10.4

Projective + Taylor $\Rightarrow$ 2-absorbing

Def. $B \subseteq \underline{A}$ is *projective* if $\forall t \in Clo(\underline{A})$ $\exists i$ $t(A, A, \ldots, A, \underset{i}{B}, A, A, \ldots A) \subseteq B$

Proposition If $B \subseteq \underline{A}$ is projective and $\underline{A}$ is Taylor, then B 2-absorbs $\underline{A}$

Proof:

- Note $B \leq \underline{A}$ $(t(B, \ldots, B) \subseteq B \ \forall t \in Clo(\underline{A}))$
- Assume $\forall t^{\in Clo(\underline{A})} \exists! i_t$ $t(A, \ldots A, \underset{i}{B}, A \ldots A) \subseteq B$

  then $Clo(\underline{A}) \longrightarrow Proj$ is a minion homo
  $t \longmapsto \pi_{i_t}$

  e.g. $s(x_1, x_2, x_3) = t(x_2, x_1, x_1, x_3, x_1, x_2)$

  $i_t = 2$ need to check $s_t = \alpha(2) = 1$

  $s(B, A, A) \subseteq t(A, B, B, A, B, A) \subseteq B$ ✓
- So $\exists t \ \exists i \neq j$ $t(A, \ldots, A, \underset{i}{B}, A, \ldots A) \subseteq B$
  and $t(A, \ldots A, \underset{j}{B}, A, \ldots \ldots A) \subseteq B$

then $B \trianglelefteq_r \underline{A}$ where

$$r(x, y) = t(*, \ldots *, \underset{j}{x}, * \ldots, \underset{i}{y}, *,)$$

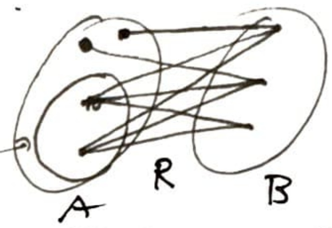
wher $*$ is x/y arbitrarily

UA $\boxed{10.5}$

$\boxed{\text{Right side has transitive op.} \Rightarrow \text{left center absorbs}}$

Def. Operation t on A is <u>transitive</u> if
$\forall a \in A \ \forall i \in [\text{arity } t] \quad t(A, \dots, A, \{a\}, A, \dots, A) = A$

Def. Left center of $R \subseteq A \times B$ is
$\{a \in A;\ \forall b \in B \ (a,b) \in R\}$



Proposition If $R \leq_{sd} \underline{A} \times \underline{B}$, $\underline{B}$ idempotent, and $t^{\underline{B}} \in Clo(\underline{B})$ is transitive, then
$\underbrace{\text{left center of } R}_{Z} \trianglelefteq_{t^{\underline{A}}} \underline{A}$.

Proof
- $\underline{B}$ idempotent $\Rightarrow Z \leq \underline{A}$ (Exercise)
- consider $\overset{\in Z}{z_1}, \overset{\in Z}{z_2}, \dots, z_{i-1}, \overset{\in A}{a}, z_{i+1}, \dots z_n$

  and $b \in B$

  want $(t^{\underline{A}}(z_1, \dots, z_{i-1}, a, z_{i+1}, \dots, z_n), b) \in R$

  - Find $c \in B$ $(a,c) \in R$ (subdirectness)
  - Find $d_1, d_2, \dots \in B$ such that
    $t^{\underline{B}}(d_1, d_2, \dots, \underset{i}{c}, d_{i+1}, \dots) = b$ (transitivity)

$$\begin{array}{ccccccccl} t^{\underline{A}}(& z_1, & z_2, & \dots, & a, & z_{i+1}, & \dots &) & \\ & R| & R| & \dots & R| & R| & & & \Rightarrow R \\ t^{\underline{B}}(& d_1 & d_2 & & c, & d_{i+1}, & \dots &) & = b \end{array}$$

$z_i \in Z$

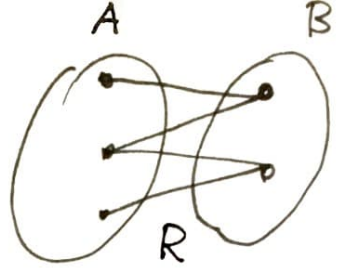
UA 10.6

transitive or projective

Proposition $\underline{A}$ finite idempotent. Then
either (i) $\exists\, t \in Clo(\underline{A})$ transitive ~~or~~
or (ii) $\exists\, \emptyset \neq B \subsetneq \underline{A}$ projective

Proof:
- Assume $\neg$(ii):
  $\forall\, \emptyset \neq B \subsetneq \underline{A}\ \ \exists\, t_B \in Clo(\underline{A})\ \ \forall i$
  $t_B(A, \dots, A, \overset{i}{B}, A, \dots A) \not\subseteq B$
  then $t_B(A, \dots A, B, A, \dots A) \supsetneq B$
  (since $\supseteq B$ always by idempotency)
- $\exists\, u\ \ \forall \emptyset \neq B \subsetneq A\ \ \forall i\ \ u(A, \dots, A, B, A, \dots, A) \supsetneq B$
  any u such that $\forall B \dots\ u^\alpha = t_B$ for some α
  (e.g. $u = t_{B_1} * t_{B_2} * \dots$
  where $B_1, B_2, \dots$ is a complete list of
  $\emptyset \neq B \subsetneq A$)
- $\exists\, t$ transitive
  $t := \underbrace{u * u * \dots * u}_{|B| \text{ times}}$
  since $\underbrace{u * \dots * u}_{j\text{-times}}(A, \dots, A, \overset{i}{\{a\}}, A, \dots A)$ has at lest j elements ($\forall a\ \forall i$)

UA 10.7



linked ⇒ left central

Def. $R \subseteq_{sd} A \times B$ is <u>linked</u>
if $(R^{-1} \circ R) \circ (R^{-1} \circ R) \circ \dots \circ (R^{-1} \circ R) = A \times A$
for a sufficient number of parenthesis

Note: • linked = connected when viewed as a bipartite graph

reflexive symmetric • $R^{-1} \circ R \subseteq (R^{-1} \circ R) \circ (R^{-1} \circ R) \subseteq \dots\dots$

Proposition: If $\exists R \lneq_{sd} \underline{A} \times \underline{B}$ linked, $\underline{A}, \underline{B}$ finite
then $\exists R \lneq_{sd} \underline{C} \times \underline{B}$ (where $\underline{C} = \underline{A}$ or $\underline{C} = \underline{B}$)
with nonempty left center

Proof:

- WLOG $R \circ R^{-1} = B^2$
  - linked ⇒ $(R \circ R^{-1}) \circ (R \circ R^{-1}) \circ \dots\dots = B^2$
  - can replace R by $R \circ R^{-1} \leq_{sd} \underline{B} \times \underline{B}$ until true
- WLOG R has empty left center
- for $D = \{d_1, \dots, d_k\} \subseteq B$ consider $S_D \subseteq B^2$

  $S_D(x,y) \equiv \exists a \in A \; R(a,x) \wedge R(a,y) \wedge R(a,d_1) \wedge \dots \wedge R(a,d_k)$
  - $S_D \leq \underline{B} \times \underline{B}$ ($\forall D \subseteq B$)
  - $S_\emptyset = B^2$ (as $R \circ R^{-1} = B^2$), $S_D = \emptyset$ (empty center)
  - take max. D such that $S_D = B^2$, $|E \setminus D| = \{1\}$
  - $S_E \neq B^2$, left center contains E, symmetric

    subdirect

✓

UA 10.8

Absorption theorem

[Barto, Kozik '10s]

Assume $\underline{A}, \underline{B}$ finite Taylor, $\exists R \lneq_{sd} \underline{A} \times \underline{B}$ linked. Then $\underline{A}$ or $\underline{B}$ has a proper absorbing subalgebra.

Proof:

- "linked $\Rightarrow$ left central" $\exists R \lneq_{sd} \underline{A} \times \underline{B}$ assume this
  (or $R \lneq_{sd} \underline{B} \times \underline{B}$) with non$\emptyset$ left center
- "transitive or projective"
  - $\underline{B}$ has a transitive $t^{\underline{B}} \in Clo(\underline{B})$
    then left center of $R \trianglelefteq_{t^{\underline{A}}} \underline{A}$
  - $\underline{B}$ has a projective $\underline{C} \lneq \underline{B}$
    then $\underline{C} \trianglelefteq \underline{B}$ by a binary operation

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so linked relations always "produce" non-trivial absorption in Taylor algebras $\rightsquigarrow$ absorption is not rare