UA $\boxed{9.1}$

RECAP

$\xi : \mathcal{A} \to \mathcal{B}$ clone homo.	$\xi : \mathcal{A} \xrightarrow{min} \mathcal{B}$ minion homo.
• $\xi(\pi^n_i) = \pi^n_i$ • $\xi(f(g_1, \dots g_m)) = \xi(f)(\xi(g_1), \dots, \xi(g_m))$	• $\xi(f(\pi^n_{i_1}, \dots, \pi^n_{i_m})) = \xi(f)(\pi^n_{i_1}, \dots, \pi^n_{i_m})$
• preservers arbitrary composition $\xi(f(g(x_1,x_2), x_3)) = \xi(f)(\xi(g)(x_1,x_2), x_3)$	• preserves minors $\xi(f(x_1,x_3,x_1,x_2)) = \xi(f)(x_1,x_3,x_1,x_2)$
• preserves identities associative op. $\mapsto$ associative op.	• preserves height 1 identities commutative op. $\mapsto$ commutative op.
• $\mathcal{A} \leq \mathcal{B}$ if $\exists \mathcal{A} \to \mathcal{B}$ $\mathcal{A} \sim \mathcal{B}$ if $\mathcal{A} \leq \mathcal{B} \leq \mathcal{A}$ $\to$ ordering	• $\mathcal{A} \overset{min}{\leq} \mathcal{B}$ if $\exists \mathcal{A} \xrightarrow{min} \mathcal{B}$ $\mathcal{A} \overset{min}{\sim} \mathcal{B}$ if $\mathcal{A} \overset{min}{\leq} \mathcal{B} \overset{min}{\leq} \mathcal{A}$ $\to$ ordering
• $\mathcal{A} \leq \mathcal{B} \Leftrightarrow \mathcal{B} \in EHSP(\mathcal{A})$	• $\mathcal{A} \overset{min}{\leq} \mathcal{B} \Leftrightarrow \mathcal{B} \in ERP(\mathcal{A})$

P_{fin} if $\mathcal{A}, \mathcal{B}$ on finite sets

relational counterparts $\rightsquigarrow$ & CSP applications

UA $\boxed{9.2}$

Taylor clones

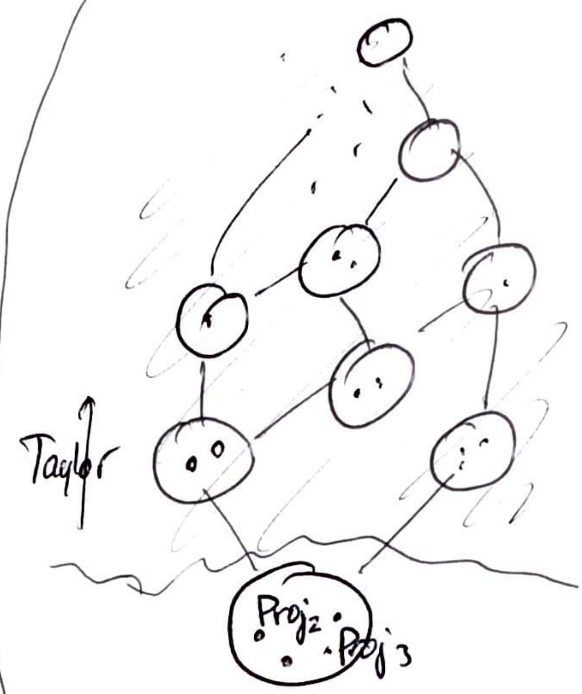
Def. Clone $\mathcal{A}$ is Taylor if

- $\mathcal{A}$ is idempotent ($\forall f \in \mathcal{A}\ \ f(x,x,\dots,x) \approx x$)
- $\mathcal{A} \not\leq \text{Proj}$

$\underline{A}$ is Taylor if $\text{Clo}(\underline{A})$ is

$\Leftrightarrow \mathcal{A} \nrightarrow \text{Proj}$

on any domain of size ≥ 2



$\leq$-ordering on Clones$/\sim$

Ex $\mathcal{A}$ idempotent & contains commutative operation $\Rightarrow \mathcal{A}$ Taylor

— " — Mal'cev operation

$\underline{A}$ idempotent & satisfies any nontrivial Mal'cev condition

not satisfiable in every algebra
= not satisfiable by projections

Recall $\mathcal{A} \leq \text{Proj}_B \Leftrightarrow \text{Proj}_B \in \text{HSP}(\mathcal{A})$

ie. $\underline{A}$ is not Taylor $\Leftrightarrow$ $\text{HSP}(\underline{A})$ contains a naked set

= ≥ 2-element algebra whose every operation is a projection

UA 9.3

Taylor operation

Def $t: A^n \to A$ is a Taylor operation if it satisfies identities of the form

$$t(x, *, *, \dots) \approx t(y, *, *, \dots)$$
$$t(*, x, *, \dots) \approx t(*, y, *, \dots)$$
$$\dots$$
$$t(*, \dots *, x) \approx t(*, \dots, *, y)$$

where $*$ are variables (wlog x/y)

Examples • Mal'cev operation is Taylor

$$m(\underline{x}, x, x) \approx m(y, y, x)$$
$$m(x, \underline{x}, x) \approx m(y, y, x)$$
$$m(x, x, \underline{x}) \approx m(x, y, \cancel{y})$$

• majority operation is Taylor

• semilattice operation (or any commutative) is T.

• constant (non-idempotent) of any arity is Taylor

(C) $\mathcal{A}$ contains a Taylor operation & is idempotent $\Rightarrow$ $\mathcal{A}$ Taylor

even $\mathcal{A} \not\leq_{min} \text{Proj}$

UA $\boxed{9.4}$

$\boxed{\text{Theorem [Taylor '7?]}}$ $\mathcal{A}$ idempotent. @

(i) $\mathcal{A} \not\leq \text{Proj}$ ($\mathcal{A}$ is Taylor) (non-trivial identities)

(ii) $\mathcal{A} \overset{\min}{\not\leq} \text{Proj}$ (non-trivial h1 identities)

(iii) $\mathcal{A}$ contains a Taylor operation

Proof: (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) ✓

minor of g

Def. $g: A^m \to A$, $\alpha: [m] \to [n]$, $g^\alpha := g(\pi^n_{\alpha(1)}, \dots, \pi^n_{\alpha(m)})$

$[m] = \{1, 2, \dots, m\}$

i.e. $g^\alpha(x_1, \dots, x_n) = g(x_{\alpha(1)}, \dots, x_{\alpha(m)})$

Ex. $g: A^3 \to A$, $\alpha: [3] \to [2]$ $g^\alpha(x_1, x_2) = g(x_2, x_1, x_2)$

$1 \mapsto 2$
$2 \mapsto 1$
$3 \mapsto 2$ ... or $\alpha = (2, 1, 2)$

$\circledcirc$ 1) $\xi: \mathcal{A} \to \mathcal{B}$ minion homomorphism $\Leftrightarrow$

preserves arity & $\forall g \dots \forall \alpha \dots$ $\xi(g^\alpha) = [\xi(g)]^\alpha$

$\circledcirc$ 2) g m-ary, $\alpha: [m] \to [n]$, $\beta: [n] \to [r]$ $(g^\alpha)^\beta = g^{\beta\alpha}$

$\circledcirc$ 3) $t: A^n \to A$ is a Taylor operation $\Leftrightarrow$

$\forall i \in [n]$ $\exists \alpha, \beta: [n] \to \dots$ $t^\alpha = t^\beta$ and $\alpha(i) \neq \beta(i)$

UA 9.5

Star-composition

Def g m-ary, f n-ary. $g*f$ ... mn-ary defined by

$$g*f(x_1,x_2,\dots,x_{mn}) := g\big(f(x_1,\dots,x_n), f(x_{n+1},\dots,x_{2n}),\dots,f(\dots,x_{mn})\big)$$

O4 $\quad g(f^{\alpha_1}, f^{\alpha_2},\dots,f^{\alpha_m}) = (g*f)^{\alpha_1{}^\frown\alpha_2{}^\frown\cdots{}^\frown\alpha_m}$

concatenation

Lemma 1 T.. finite set of idempotent operations on A.
Then $\exists F\in \mathrm{Clo}(T)\ \forall f\in T\ F^{\alpha}=f$ for some α

Proof:
- Say $T=\{f,g\}$
- $F := g*f$
- $F(x_1,x_1,\dots,x_1,x_2,x_2,\dots,x_2,\dots x_m,\dots x_m) = g(x_1,\dots,x_n)$
  (ie $F^{(1,1,\dots,1,2,2,\dots,2,\dots,m,\dots,m)} = g$)
- $F(x_1,x_2,\dots,x_n,x_1,x_2,\dots,x_n,\dots,x_1,x_2,\dots,x_n)$
  $= f(x_1,x_2,\dots,x_n)$
  (ie $F^{(1,2,3,\dots,n,1,2,\dots,n,\dots)} = f$)

in general $T=\{f_1,f_2,\dots f_i\}$ $\quad F := f_1 * f_2 * \dots * f_i$

VA $\boxed{9.6}$

(i) $\Rightarrow$ (ii) $\mathcal{A} \not\equiv Proj \Rightarrow \mathcal{A} \overset{min}{\not\equiv} Proj$

$\boxed{\text{Will prove more:}}$ $\forall\, \xi : \mathcal{A} \xrightarrow{min} Proj$ <u>is</u> a clone homomorphism

- Take $\xi : \mathcal{A} \xrightarrow{min} Proj$

- Lemma 2 $\quad t \in \mathcal{A}_m,\ s \in \mathcal{A}_n \quad \xi(\overbrace{t * s}^{h}) = \xi(t) * \xi(s)$

  - say $\xi(t) = \pi^m_i$, $\xi(s) = \pi^n_j$, $\xi(h) = \pi^{mn}_k$

    index projections from 0 (π^m_0 is the 1st proj.)

  - RHS $= \pi^{mn}_{ni+j}$

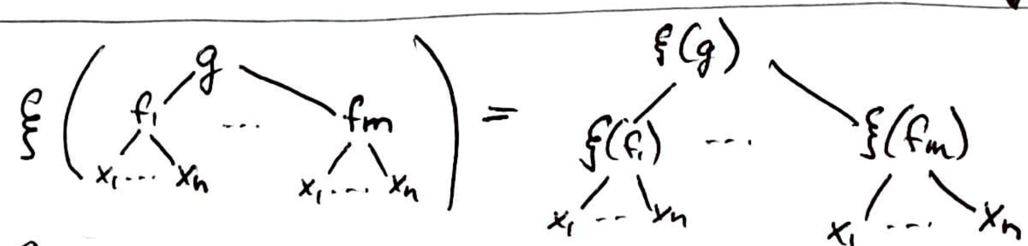
  - $[\xi(h)]^{(1,2,\dots,n,1,2,\dots,n,\dots)} \overset{\circledcirc 1}{=} \xi(h^{(1,2,\dots,n,1,2,\dots,n,\dots)})$

    $= \xi(s) \Rightarrow k \bmod n = j$

  - $[\xi(h)]^{(1,1,\dots,1,2,2,\dots,\dots)}$ ...... $\Rightarrow k \operatorname{div} n = i$

  $\Big\}\ k = ni + j$ ✓

- want



- take $\overset{\in \mathcal{A}}{F}$ from Lemma 1 such that $F^{\alpha_i} = f_i$ $(\alpha_i : [\text{arity } F] \to [n])$

$$LHS = \xi(g(f_1,\dots,f_m)) = \xi(g(F^{\alpha_1},\dots,F^{\alpha_m})) \overset{\circledcirc 4}{=}$$

$$\xi\left((g * F)^{\alpha_1 \frown \alpha_2 \dots \frown \alpha_m}\right) \overset{\circledcirc 1}{=} \left(\xi(g * F)\right)^{\alpha_1 \frown \alpha_2 \dots \frown \alpha_m} \overset{\text{Lemma 2}}{=} \left(\xi(g) * \xi(F)\right)^{\alpha_1 \frown \alpha_2 \frown \dots}$$

$$\overset{\circledcirc 4}{=} \xi(g)\left((\xi(F))^{\alpha_1},\dots,(\xi(F))^{\alpha_m}\right) \overset{\circledcirc 1}{=} \xi(g)\left(\xi(F^{\alpha_1}),\dots,\xi(F^{\alpha_m})\right)$$

$$= \xi(g)\left(\xi(f_1),\dots,\xi(f_m)\right) = RHS$$

UA $\boxed{9.7}$

we need to select "important coordinate" for each $t \in \mathcal{A}$ so that the selection goes well with minors

$(ii) \Rightarrow (iii)$ no Taylor $\Rightarrow$ $\mathcal{A} \xrightarrow{min} \text{Proj}$

$$\xi(t^\alpha) = [\xi(t)]^\alpha$$

- Assume $\mathcal{A}$ doesn't contain Taylor operation

Step 1 $\forall t \in \mathcal{A} \ \exists \underset{\text{arity preserving}}{\xi : \mathcal{A} \to \text{Proj}} \ \forall \alpha \dots \quad \xi(t^\alpha) = [\xi(t)]^\alpha$

- t not Taylor $\overset{\text{ö}3}{\Longrightarrow}$ $\exists i \ \forall \alpha, \beta \quad t^\alpha = t^\beta \Rightarrow \alpha(i) = \beta(i)$
- define $\xi(t^\alpha) := \pi_{\alpha(i)}$ $\quad (\Rightarrow \xi(t) = \pi_i)$
  - makes sense
  - $\xi(t^\alpha) = \pi_{\alpha(i)} = [\xi(t)]^\alpha$

Step 2 $\forall T \subseteq_{fin} \mathcal{A} \ \exists \xi \ \forall t \in T \ \forall \alpha \dots \ \xi(t^\alpha) = [\xi(t)]^\alpha$

- Lemma 1 .. $\exists F \ \forall t \in T \quad t = F^\beta$ for some β
- $\xi :=$ from Step 1 for F
- $\xi(t^\alpha) = \xi((F^\beta)^\alpha) \overset{\text{ö}2}{=} \xi(F^{\alpha\beta}) \overset{\text{Step 1}}{=} [\xi(F)]^{\alpha\beta} =$

$$\overset{\text{ö}2}{=} ([\xi(F)]^\beta)^\alpha \overset{\text{Step 1}}{=} (\xi(F^\beta))^\alpha = (\xi(t))^\alpha$$

Step 3 $\exists$ minion homo $\xi : \mathcal{A} \to \text{Proj}$ – compactness argument

0-ary symbols: π_i^n, $\xi(t)$

sentences: $n \in \mathbb{N}$ • $\pi_1^n \neq \pi_2^n \neq \dots$

$\forall t \in \mathcal{A}$ arity n • $\xi(t) = \pi_1^n \vee \xi(t) = \pi_2^n \vee \dots$

$\forall t \ \forall \alpha \ s := t^\alpha$ "$\xi(s) = (\xi(t))^\alpha$" • $\xi(t) = \pi_i^n \Rightarrow \xi(s) = \pi^n_{\alpha(i)}$, ...

$\forall$ finite subset has a model (Step 2)

$\Rightarrow \exists$ model

UA $\boxed{9.8}$

<u>We proved</u>: $\mathcal{A}$ idempotent:

- $\mathcal{A} \nleq \mathrm{Proj}$
- $\mathcal{A} \overset{min}{\nleq} \mathrm{Proj}$
- $\mathcal{A}$ contains a Taylor op.
- ~~$\mathcal{A}$~~ $\mathrm{Proj}_2 \notin HSP(\mathcal{A})$

in general

in general

<u>Corollary</u> $\mathcal{A}$ finite (not necessarily idempotent)

(i) $\mathcal{A} \overset{min}{\nleq} \mathrm{Proj}$ (this is what interests us for CSP)

(ii) $\mathcal{A}$ contains a Taylor operation

Proof: (ii) → (i) ✓

(i) → (ii)
- $\exists \mathcal{B}$ idempotent $\mathcal{A} \sim \mathcal{B}$
- $\mathcal{B} \overset{min}{\nleq} \mathrm{Proj} \Rightarrow \mathcal{B}$ contains a Taylor op.
- $\mathcal{B} \leq \mathcal{A} \Rightarrow \mathcal{A}$ contains a Taylor op. (since Taylor identities are height 1)

<u>But</u>
- for finite (non-idempotent) $\mathcal{A} \overset{min}{\leq} \mathrm{Proj} \not\Rightarrow \mathcal{A} \leq \mathrm{Proj}$
- for infinite (~~id~~ —"—) $\mathcal{A} \overset{min}{\nleq} \mathrm{Proj} \not\Rightarrow \mathcal{A}$ contains a Taylor

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Viewpoint via important coordinates/variables

$\S$: $\mathcal{A} \overset{min}{\leq} \mathrm{Proj} \Leftrightarrow \exists I : \mathcal{A} \to \mathbb{N}$ such that

- $I(f) \in \{1, 2, \dots, \text{arity } f\}$ $(\forall f \in \mathcal{A})$
- $I(f^\alpha) = \alpha(I(f))$

$(\forall f \in \mathcal{A}\ \forall \alpha : [\text{arity } f] \to [\dots])$

(UA) $\boxed{9.9}$

$\boxed{\text{Theorem}}$ $\mathbf{A}$ idempotent finite @

- $\mathbf{A} \notin \mathrm{Proj}_2$ (Taylor)
- $\mathbf{A} \overset{\min}{\notin} \mathrm{Proj}_2$

- $\mathrm{Proj}_2 \in \mathsf{HSP}(\mathbf{A})$
- $\mathrm{Proj}_2 \in \mathsf{HS}\cancel{\mathsf{P}}(\mathbf{A})$ [Bulatov '00s]

practical

- $\mathbf{A}$ contains a Taylor operation [Taylor '70s]
- $\mathbf{A}$ contains a weak near-unanimity operation

  $w(y,x,x,\dots x) \approx w(x,y,x,\dots x) \approx \dots \approx w(x,x,\dots,x,y)$

  (arity ≥ 2) [McKenzie, Maróti '00s]

- $\mathbf{A}$ contains a p-ary cyclic operation for every prime $p > |A|$ [Barto, Kozik '10s]

  $c(x_1, \dots, x_p) \approx c(x_2, \dots, x_p, x_1)$

- $\mathbf{A}$ contains a 6-ary Siggers operation [Siggers '10s]

  $s(x,y,x,z,y,z) \approx s(y,x,z,x,z,y)$

- $\mathbf{A}$ contains a 4-ary Siggers operation

  $s(r,a,r,e) \approx s(a,r,e,a)$ [Kearnes, Marković, McKe

we will prove this

UA 9.10

Theorem $\mathbb{A}$ idempotent $\circlearrowleft$ @

- $\mathbb{A} \notin$ Proj
- $\mathbb{A} \overset{min}{\notin}$ Proj
- $\mathbb{A}$ has a Taylor operation

- $\mathbb{A}$ has an Olšák operation [Olšák '16]

$$o(x, x, y, y, y, x)$$
$$\approx o(x, y, x, y, x, y)$$
$$\approx o(y, x, x, x, y, y)$$

This theorem & some items in the previous one uses the concept of absorbing subuniverses

the rest of the course