UA 8.1

RECAP

$\mathcal{R}$... finite set of relation on a finite set A ($|A| \geq 2$)

CSP($\mathcal{R}$)

INPUT: pp-sentence over $\mathcal{R}$
($\exists x, y, u, \dots \; R(x,u,x) \wedge S(u,y) \wedge \dots$)

QUESTION: true (in $\mathcal{R}$)?

If $\mathcal{S}$ is pp-definable from $\mathcal{R}$

$\Leftrightarrow$ $\mathrm{RelClo}(\mathcal{S}) \subseteq \mathrm{RelClo}(\mathcal{R})$

$\Leftrightarrow$ $\mathrm{Pol}(\mathcal{R}) \subseteq \mathrm{Pol}(\mathcal{S})$

Then $\mathrm{CSP}(\mathcal{S}) \leq \mathrm{CSP}(\mathcal{R})$

$\Rightarrow$ If $\mathrm{Pol}(\mathcal{R})$ = projections, then $\mathrm{CSP}(\mathcal{R})$ NP-complete

This NP-completeness condition is insufficient

TODAY clone & minion homomorphisms

$\rightsquigarrow$ 2 ways to compare clones (other than $\subseteq$)

$\rightsquigarrow$ better NP-completeness condition

UA 8.2

Clone homomorphism

Def. $\mathcal{A}, \mathcal{B}$ clones. A mapping $\xi : \overset{\text{on } A}{\mathcal{A}} \to \overset{\text{on } B}{\mathcal{B}}$ is a <u>clone homomorphism</u> if it

- preserves arities
- preserves projections $\left(\xi(\underset{\text{on } A}{\pi^n_i}) = \underset{\text{on } B}{\pi^n_i} \right)$
- preserves composition $\left(\begin{array}{l} \xi(g(f_1, \dots, f_m)) = \\ \xi(g)(\xi(f_1), \dots, \xi(f_m)) \end{array} \right)$

Write $\mathcal{A} \leq \mathcal{B}$ if $\exists$ homo $\mathcal{A} \to \mathcal{B}$

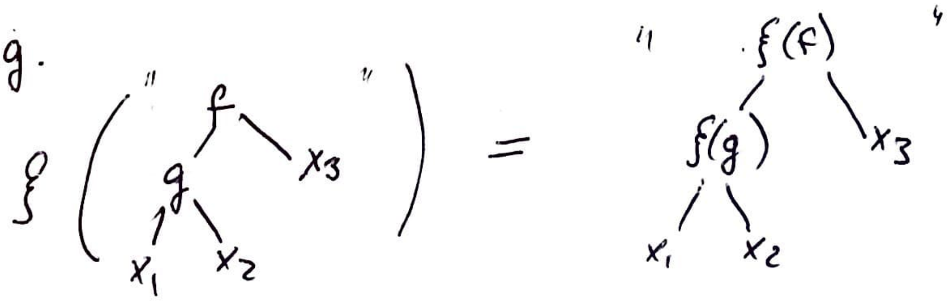
$\mathcal{A} \sim \mathcal{B}$ if $\mathcal{A} \leq \mathcal{B} \leq \mathcal{A}$

! mapping operations $\to$ operations

not ~~$A \to B$~~

(!) clone homomorphisms preserve "arbitrary composition"

e.g.



the operation

$(x_1, x_2, x_3) \mapsto f(g(x_1, x_2), x_3)$

UA $\boxed{8.3}$

$\cong$

⓪ $\xi: \mathcal{A} \to \mathcal{B}$ is a clone homomorphism
$\iff$ ξ preserves arity and "identities"

e.g. • if $m \in \mathcal{A}_3$ satisfies $m(x,x,y) \approx y$ ($\forall x,y \in A$)
then $\xi(m) \in \mathcal{B}_3$ satisfies $\xi(m)(x,x,y) \approx y$ ($\forall x,y \in B$)

• if $f,g,h \in \mathcal{A}_2$ satisfy $f(x,g(y,x)) \approx h(x,y)$
then $\xi(f)(x, \xi(g)(y,x)) \approx \xi(h)(x,y)$

$(\Rightarrow)$ If $\mathcal{A}$ satisfies a Mal'tsev condition (M) e.g. "there is a Mal'tsev operation"
then so ~~des~~ does $\mathcal{B}$

Examples

• $\mathcal{A} \subseteq \mathcal{B}$ $\quad \xi: \mathcal{A} \to \mathcal{B}$, $f \mapsto f$ $\quad$ write $\mathcal{B} \in E(\mathcal{A})$ expansion

• $\mathcal{A} = Clo(\underline{A})$, $\underline{B} \leq \underline{A}$, $\mathcal{B} = Clo(\underline{B})$ $\quad \xi: \mathcal{A} \to \mathcal{B}$, $f \mapsto f\restriction_B$ $\quad (t^{\underline{A}} \mapsto t^{\underline{B}})$
$\mathcal{B} \in S(\mathcal{A})$

• $\mathcal{A} = Clo(\underline{A})$, $\underline{B} = \underline{A}^X$, $\mathcal{B} = Clo(\underline{B})$ $\quad \xi: \mathcal{A} \to \mathcal{B}$, $f \mapsto f^X$ $\quad (t^{\underline{A}} \mapsto t^{\underline{B}})$
$\mathcal{B} \in P(\mathcal{A})$

• $\mathcal{A} = Clo(\underline{A})$, ~~$\underline{B} \in H(\underline{A})$~~ $\underline{B} = \underline{A}/\sim$, $\mathcal{B} = Clo(\underline{B})$ $\quad \xi: \mathcal{A} \to \mathcal{B}$, $f \mapsto$ "$f/\sim$" $\quad (t^{\underline{A}} \mapsto t^{\underline{B}})$
$\mathcal{B} \in H(\mathcal{A})$

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(0) $\underline{A}, \underline{B}$ of the same signature. $\xi : \mathcal{A} \to \mathcal{B}$

$t^{\underline{A}} \mapsto t^{\underline{B}}$

makes sence $\Leftrightarrow$ $Id(\underline{A}) \subseteq Id(\underline{B})$.

If it makes sense, it is a clone homomorphism

Birkhoff's HSP Theorem for Clones / $\mathcal{A}, \mathcal{B}$ clones. TFAE

(i) $\mathcal{A} \leq \mathcal{B}$

(ii) $\mathcal{B} \in EHSP(\mathcal{A})$

finite powers enough if A, B finite

(ii) $\to$ (i) by examples

(i) $\to$ (ii)

- take $\underline{A} = (A; g_1, g_2, \dots)$ s.t. $\mathcal{A} = Clo(\underline{A})$

  Let $\underline{C} = (B; \xi(g_1), \xi(g_2), \dots)$. Note $Clo(\underline{C}) \subseteq \mathcal{B}$

- enough to show $\underline{C} \in HSP(\underline{A})$ ← Birkhoff directly

Direct proof for finite $B = \{1, 2, \dots, n\}$:

$$S(\underline{A}^{A^n}) \in SP(\underline{A})$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \\ \underline{Clo_n(\underline{A})} & \longrightarrow & \underline{B} \\ f & \longmapsto & (\xi(f))(1, 2, \dots, n) \end{array}$$

- onto because of projections
- homo because of definitions

UA

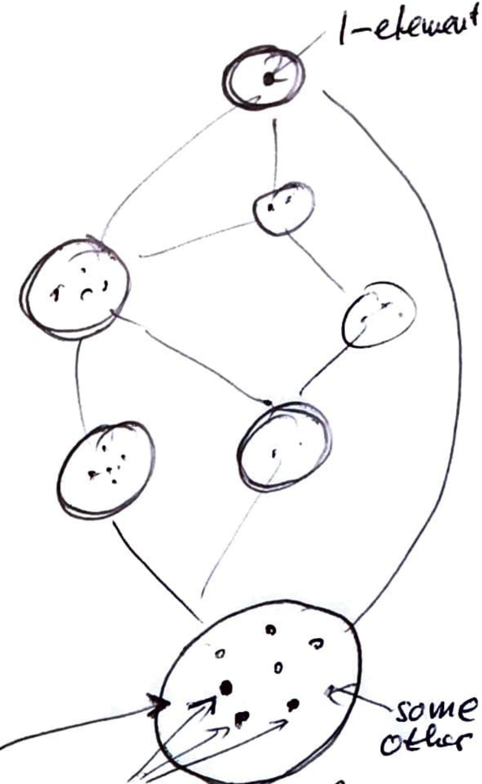
8.5

$\leq$ ... preorder on the class CL of all clones

$\leq$ ... ordering on $CL/\sim$

in fact a lattice

$\rightsquigarrow$ <u>interpretability lattice</u>



$\mathcal{R}$ .. set of relations on A

$\mathcal{A} = Pol(\mathcal{R})$

$\mathcal{S}$ .. set of relations on B

$\mathcal{B} = Pol(\mathcal{S})$

previously

~~Pol~~ $\mathcal{A} \subseteq \mathcal{B}$

$\Leftrightarrow \mathcal{B} \in E(\mathcal{A})$

$\Leftrightarrow \mathcal{S}$ pp-definable from $\mathcal{R}$

$\to CSP(\mathcal{S}) \leq CSP(\mathcal{R})$

better:

$\mathcal{A} \leq \mathcal{B}$

$\Leftrightarrow \mathcal{B} \in EHSP(\mathcal{A})$

$\Leftrightarrow \mathcal{S}$ pp-interpretable in $\mathcal{R}$

$\to CSP(\mathcal{S}) \leq CSP(\mathcal{R})$

$\Rightarrow$ complexity of $CSP(\mathcal{R})$ depends only on position of $Pol(\mathcal{R})$ in the $\leq$ order

$\Rightarrow$ if $Pol(\mathcal{R}) \to Proj$, then $CSP(\mathcal{R})$ is NP-complete

UA 8.6

Minion homomorphism

Def. $\mathcal{A}, \mathcal{B}$ clones. A mapping $\xi : \mathcal{A} \to \mathcal{B}$ is a <u>minion homomorphism</u> (or <u>h1-clone homomorphism</u>) if it

- preserves arities
- preserves composition with projection, i.e.
  $$\xi(f(\pi^n_{i_1}, \pi^n_{i_2}, \dots, \pi^n_{i_m})) = \xi(f)(\pi^n_{i_1}, \dots, \pi^n_{i_m})$$

Write $\mathcal{A} \overset{min}{\leq} \mathcal{B}$ if $\exists$ minion homo $\mathcal{A} \overset{min}{\to} \mathcal{B}$

$\mathcal{A} \overset{min}{\sim} \mathcal{B}$ if $\mathcal{A} \overset{min}{\leq} \mathcal{B} \overset{min}{\leq} \mathcal{A}$

(i) presererves composition with projections = preserves minors

e.g. $\xi(\underbrace{f(x_1, x_3, x_1, x_2)}_{\text{minor of } f}) = \xi(f)(x_1, x_3, x_1, x_2)$

(ii) ~~pr~~ = preserves height 1 identities

✓ • $f(x_1, x_2, x_4, x_1) \approx g(x_2, x_1) \Rightarrow \xi(f)(x_1, x_2, x_4, x_1) = \xi(g)(x_2, x_1)$

height 1 ... both sides are op.(variables)

✗ $\underbrace{f(x_1, x_2, x_2) \approx x_1}_{\text{not heigh 1}} \not\Rightarrow \xi(f)(x_1, x_2, x_2) \approx x_1$

✗ $\underbrace{f(f(x_1, x_2), x_3)}_{\text{not heigh 1}} \approx \dots \not\Rightarrow$

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see homework

Examples

E, $\underbrace{\text{H, S}}$, P

common generalization – R reflection

<u>Version of Birkhoff</u>: $\mathcal{A} \overset{\min}{\leq} \mathcal{B} \iff \mathcal{B} \in ERP(\mathcal{A})$

$\mathcal{A} = Pol(\mathbb{R})$ $\mathcal{B} = Pol(\mathbb{S})$

previously	better	still better
$\mathcal{A} \leq \mathcal{B}$	$\mathcal{A} \leq \mathcal{B}$	$\mathcal{A} \overset{\min}{\leq} \mathcal{B}$
$\iff \mathcal{B} \in E(\mathcal{A})$	$\iff \mathcal{B} \in EHSP(\mathcal{A})$	$\iff \mathcal{B} \in ERP(\mathcal{A})$
$\iff \mathbb{S}$ pp-def. from $\mathbb{R}$	$\iff \mathbb{S}$ pp-interpretable in $\mathbb{R}$	$\iff \mathbb{S}$ pp-constructible
$\Rightarrow CSP(\mathbb{S}) \leq CSP(\mathbb{R})$	$\Rightarrow$ —"—	$\Rightarrow$ —"—

$\Rightarrow$ complexity of $CSP(\mathbb{R})$ depends only on position of $Pol(\mathbb{R})$ in $\overset{\min}{\leq}$ order

$\Rightarrow$ if $Pol(\mathbb{R}) \xrightarrow{\min} Proj$, then $CSP(\mathbb{R})$ NP-complete

THEOREM [Bulatov'17, Zhuk'17] Otherwise $CSP(\mathbb{R})$ in P

~~the~~ clones on 2-elements/$\overset{\min}{\sim}$ ordered by $\overset{\min}{\leq}$

[Bodirsky, Vucaj'20]