UA2 $\boxed{7.1}$

WAS

- abelianess, commutator theory
- equational theories
  - term rewriting
  - finite base

NOW

Taylor algebras

one motivation
- Constraint Satisfaction Problems
  - significant application of UA to TCS

more on signature-free UA
- clone homomorphisms, minion homomorphisms

classic
- Taylor algebras, Taylor's theorem

one of the new tools
- Absorption theory
  - absorption theorem
  - finite + Taylor + abelian $\Rightarrow$ affine
  - loop lemma + applications

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CSPs Constraint Satisfaction Problems over fixed finite templates

$\mathcal{R}$ ... finite set of relations on a finite set A

CSP($\mathcal{R}$) ... CSP over $\mathcal{R}$ → template

INPUT: $\underbrace{\text{primitive positive}}_{\exists, \wedge, =,\ R\in\mathcal{R}}$ $\underbrace{\text{sentence}}_{\text{no free variables}}$ over $\mathcal{R}$

QUESTION: is it true?

<u>Example</u> $\mathcal{R} = \{R, S\}$, $R \subseteq A$, $S \subseteq A^2$

input e.g. $\underbrace{\exists x, y, z, u}_{\text{in } A}$ $R(y) \wedge S(x, z) \wedge (y = z) \wedge S(x, x) \wedge R(u) \wedge S(u, z)$

WLOG quantifiers first

constraints

- one computational problem for each $\mathcal{R}$
- computational complexity?
  - always in NP
  - sometimes in P, e.g. if $\mathcal{R}$ contains only unary relations
  - sometimes NP-complete *
  - "=" can be ignored for our purposes
- many variants: counting, optimization, approximation, infinite A, promise problems, ...

UA2 Z. 3

Example: 3-coloring

3-COLOR

INPUT: graph

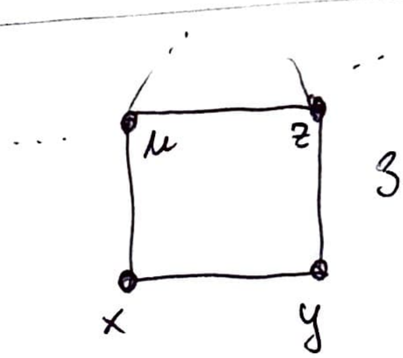
QUESTION: is it 3-colorable?

i.e. $\exists$ mapping c: vertices $\to \{1,2,3\}$
such that $a - b \Rightarrow c(a) \neq c(b)$?

- $A = \{1,2,3\}$
- $R = \{\neq\}$ $\neq \; := \{(a,b) \in A^2;\ a \neq b\}$

3-COLOR $\sim$ CSP(R)

equivalent modulo polynomial-time reductions
here: essentially the same problem



3-colorable $\iff$ A
$\exists x, y, z, u$
$(x \neq y) \wedge (y \neq z) \wedge (z \neq u) \wedge (x \neq u)$

---

complexity

- 3-COLOR, 4-COLOR, ... NP-complete
- 2-COLOR, 1-COLOR P

UA 2   7.4

Example: 3-satisfiability

3-SAT

INPUT: Boolean formula in conjunctive form, each conjunct is a disjunction of 3 literals; literal is
- variable
- negated variable

QUESTION: is it satisfiable?

- $A = \{0,1\}$
- $\mathcal{R} = \{R_{000}, R_{001}, \dots, R_{111}\}$ $R_{ijk} := \{0,1\}^3 \setminus \{(i,j,k)\}$

3-SAT $\sim$ CSP($\mathcal{R}$)

$(x \vee y \vee z) \wedge (\neg x \vee u \vee \neg y) \wedge (u \vee \neg z \vee \neg u)$

is satisfiable

$$\Longleftrightarrow$$

$$\exists x, y, z, u \quad R_{000}(x,y,z) \wedge R_{101}(x,u,y) \wedge R_{011}(u,z,u)$$

---

complexity

- 3-SAT, 4-SAT, ..   NP-complete (very well known)
- 2-SAT   P

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Example: ~~Bla~~ Linear equations

3-LIN-p

INPUT: system of linear equations over $\mathbb{Z}_p$
each equation has 3 variables

QUESTION: is it solvable?

- $A = \{0, 1, \dots, p-1\}$
- $R = \{T_{0000}, T_{0001}, \dots, T_{p-1,p-1,p-1,p-1}\}$ where
  $T_{abcd} := \{(x,y,z) \in A^3;\ ax + by + cz = d\}$

3-LIN-p $\sim$ CSP(R)

$2x + 3y + z = 6$ solvable
$y + 2u + x = 3$

$\Leftrightarrow\ \exists x, y, z, u\ \ T_{2316}(x,y,z) \wedge T_{1213}(y,u,x)$

---

complexity • 3-LIN-p, ... P

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Short history

- [Schaefer'78]
  if $|A|=2$ $\quad CSP(\mathbb{R}) \begin{cases} P \\ NP\text{-complete} \end{cases}$ $\quad$ dichotomy + exact description

- [Hell, Nešetřil'90]
  $\mathbb{R} = \{R\}$ R binary, symmetric
  $CSP(\mathbb{R}) \begin{cases} P \\ NP\text{-complete} \end{cases}$

- [Feder, Vardi '93,98]
  <u>Conjecture</u>: $\forall \mathbb{R}$ $CSP(\mathbb{R}) \begin{cases} P \\ NP\text{-complete} \end{cases}$
  "Dichotomy conjecture"

- [Jeavons'98, Bulatov, Jeavons, Krokhin'00]
  $\rightsquigarrow$ UA, precise borderline conjectured
  "Algebraic dichotomy conjecture"

⋮

many results

⋮

- [Bulatov'17, Zhuk'17]
  Algebraic dichotomy conjecture confirmed

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CSPs & pp-definitions

$\mathcal{R}, \mathcal{S}$ ... finite sets of relations on A

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(0) [Schaefer] If $\mathcal{S}$ is pp-definable from $\mathcal{R}$,
(i.e., every relation in $\mathcal{S}$ is pp-definable from $\mathcal{R}$)
then $CSP(\mathcal{S}) \leq CSP(\mathcal{R})$

polynomial time many-one reduction

Prf: want mapping inputs of $CSP(\mathcal{S}) \to$ inputs of $CSP(\mathcal{R})$
such that YES instances are mapped to YES instances
and NO —— " —— NO instances

the mapping input $\varphi \mapsto$ replace relations in $\mathcal{S}$ by their pp-definitions in $\mathcal{R}$

$\Rightarrow$ If $\mathrm{Relclo}(\mathcal{S}) \subseteq \mathrm{Relclo}(\mathcal{R})$,
then $CSP(\mathcal{S}) \leq CSP(\mathcal{R})$

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CSPs & clones

$\mathcal{R}, \mathcal{S}$ .. finite sets of relations on A

!!! [Jeavons] If $Pol(\mathcal{R}) \subseteq Pol(\mathcal{S})$, then
$$CSP(\mathcal{S}) \leq CSP(\mathcal{R})$$

because $Pol(\mathcal{R}) \subseteq Pol(\mathcal{S}) \iff Relclo(\mathcal{R}) \supseteq Relclo(\mathcal{S})$

$\Rightarrow$ the complexity of $CSP(\mathcal{R})$ depends only on ~~CSP~~ $Pol(\mathcal{R})$

$\Rightarrow$ if $|A| \geq 2$ & $Pol(\mathcal{R})$ = projections, then $CSP(\mathcal{R})$ is NP-complete
(since $\exists$ NP-complete CSP on every domain of size ≥ 2)

- it's not the only reason for hardness (see 3-COLOR)
  → need more reductions → clone homomorphisms, minion homomorphisms } modern formulation