UA2 $\boxed{6.1}$

RECAP

Σ ... finite signature

$\mathcal{E}$ is a base of identities in $\mathrm{Id}(\mathcal{V})$

$\mathcal{V}$ is finitely based if
$\exists$ finite $\mathcal{E}$ such that $\mathcal{V} \models s \approx t$ iff $\mathcal{E} \vdash s \approx t$
$\Leftrightarrow$ finitely axiomatizable

$\underline{A}$ finite. @

- $HSP(\underline{A})$ finitely based
- $\exists n$ $\{$n-variable identities$\}$ is a base

$\exists$ finite algebras $\underline{A}$ that are not finitely based

TODAY

A finite basis result

- does not cover a "large" class of varieties
- its successors do

UA 2 $\boxed{6.2}$

THEOREM [McKenzie'78] Suppose $\mathcal{V}$ has
(1) definable principal congruences (DPC) and
(2) only finitely many SIs (up to isomorphisms), all of them finite.
Then $\mathcal{V}$ is finitely based

---

(2) satisfied if e.g. $\mathcal{V} = HSP(\underline{A})$ and $\mathcal{V}$ is CD
($\underline{A}$ finite)
(Jónsson's lemma $\to$ all SIs are in $HS(\underline{A})$)
e.g. $\underline{A}$ finite lattice

(1) "almost never" satisfied but weaker conditions "quite often" (proofs more difficult)

recall
- lattices have majority term
  $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$
- $\mathcal{V}$ has a majority term
  $\Rightarrow \mathcal{V}$ is CD (practicals last semester)

also recall: varieties determined by SIs

UA2 $\boxed{6.3}$

$\boxed{\text{Def.}}$ $\mathcal{V}$ has <u>DPC</u> (definable principal congruences) if $\exists$ formula $\underbrace{\varphi(x,y,a,b)}_{\text{free variables}}$ equivalent in $\mathcal{V}$ to "$(x,y) \in Cg(a,b)$", i.e.

$$\forall \underline{A} \in \mathcal{V} \ \forall x,y,a,b \in A \ \ (x,y) \in Cg_{\underline{A}}(a,b) \Leftrightarrow \underline{A} \models \varphi(x,y,a,b)$$

$$\Leftrightarrow \forall A \in \mathcal{V} \ \forall a,b \in A \quad \varphi\text{-}Cg^{\underline{A}}(a,b) = Cg^{\underline{A}}(a,b)$$

where $\varphi\text{-}Cg^{\underline{A}}(a,b) := \{(x,y) \in A^2;\ \underline{A} \models \varphi(x,y,a,b)\}$

"what φ thinks is $Cg(a,b)$"

$\boxed{\text{Examples}}$

- ✓ $\mathcal{V}$ = commutative rings has DPC $\quad \varphi \equiv \exists z \ x-y = z\cdot(a-b)$
- ✓ $\mathcal{V}$ = abelian groups of exponent 1000 has DPC

  φ: $\underbrace{(x-y=0) \vee (x-y=a-b) \vee (x-y=b-a) \vee (x-y = 2(a-b)) \vee \ldots}_{\text{"}x-y \text{ is in the cyclic subgroup gen. by } a-b\text{"}}$

- ✓ $\left.\begin{array}{l}\mathcal{V} = \text{distributive lattices} \\ \mathcal{V} = \text{semilattices}\end{array}\right\}$ have DPC

- ✗ $\mathcal{V}$ = abelian groups does not have DPC

UA2 $\boxed{6.4}$

$\boxed{\mathcal{V} = \text{abelian groups does not have DPC}}$

φ_n: $(x-y=0) \vee (x-y=a-b) \vee \dots \vee (x-y=n(a-b)) \vee (x-y=n(b-a))$

clearly $\forall \underline{A} \in \mathcal{V}$ $\forall x,y,a,b$ $(x,y) \in Cg(a,b)$ iff $\varphi_1(x,y,a,b) \vee \varphi_2(x,y,a,b) \vee \dots$

assume $\varphi(x,y,a,b)$ defines principal congruences

so $\varphi(x,y,a,b) \xleftrightarrow{\text{within } \mathcal{V}} \underbrace{\varphi_1(x,y,a,b) \vee \varphi_2(x,y,a,b) \vee \dots}_{\text{infinite disjunction}}$

Then $\varphi(x,y,a,b) \longleftrightarrow$ finite subdisjunction
which is false (see $\mathbb{Z}$)

Why?

$\Sigma = \{+, -, 0, \underbrace{c, d, e, f}_{\text{new 0-ary symbols}}\}$

sentences

ab. group axioms
$\varphi(c,d,e,f)$
$\neg\varphi_1(c,d,e,f)$
$\neg\varphi_2(c,d,e,f)$
$\vdots$
} have no model

compactness for logic

$\Rightarrow$ some finite set of sentences has no model

$\Rightarrow$
ab. group axioms
$\varphi(c,d,e,f)$
$\neg\varphi_{n_1}(c,d,e,f)$
$\vdots$
$\neg\varphi_{n_k}(c,d,e,f)$
} no model $\Leftrightarrow$ $\varphi(x,y,a,b) \rightarrow \varphi_{n_1}(x,y,a,b) \vee \dots \vee \varphi_{n_k}(x,y,a,b)$

UA2 $\boxed{6.5}$

We will need "nice" formulas defining principal congruences

$\boxed{\text{Def}}$ $\varphi(x,y,a,b)$ is <u>conservative</u> if $\forall \underline{B}$ in the signature $\forall a,b \in B$ $\varphi\text{-}Cg_{\underline{B}}(a,b) \subseteq Cg_{\underline{B}}(a,b)$

i.e. "$\varphi(x,y,a,b) \Rightarrow (x,y) \in Cg(a,b)$"

Examples

- $\mathcal{V}$ = commutative rings
  - $\varphi = \exists z \;\; x-y = z\cdot(a-b)$ is not conservative (take $\underline{B}$ such that $\forall u,v\in B$ $u-v=u$, $u\cdot v = u$)
  - $\varphi \equiv \exists z,r \;\; (x = r\cdot a + z) \wedge (y = r\cdot b + z)$ is conservative (and defines principal congruences in $\mathcal{V}$ like ✓)
- more complex conservative formula $(\Sigma \supseteq \{\cdot, +, *\})$

$$\exists z_1, z_2, z_3 \;\; (x = z_1 \cdot a + z_2) \wedge (z_1 \cdot b + z_2 = (z_3 * b) * z_4) \wedge ((z_3 * a) * z_4 = y)$$

$\overset{!!!}{\text{(o)}}$ disjunction of conservative formulas is conservative

$\overset{!!!}{\text{(o)}}$ For every signature "$(x,y)\in Cg(a,b)$" is equivalent to infinite disjunction of conservative formulas

(why?) $Cg^{\underline{A}}(a,b)$ is the transitive closure of the subalgebra of $\underline{A}^2$ generated by the equivalence $|ab|\cdot|\cdot|\cdot|\cdot|\cdot|\ldots$

– this can be written (Exercise!) as infinite disjunction of conservative formulas (\& see ↓)

UA2 $\boxed{6.6}$

$\rightsquigarrow$ $\mathcal{V}$ has DPC, then $(x,y) \in Cg(a,b)$ is equivalent to $\varphi(x,y,a,b)$ (within $\mathcal{V}$)
and this is equivalent to infinite $\bigvee$ of conservative formulas

$\Longrightarrow$ $(x,y) \in Cg(a,b)$ is equivalent to a conservative formula

More generally:

$\boxed{\text{Theorem}}$ Assume $\mathcal{V}$ is axiomatizable and

$$\psi(x_1,\dots,x_n) \leftrightarrow \underbrace{\psi_1(x_1,\dots,x_n) \vee \psi_2(x_1,\dots,x_n) \vee \dots\dots}_{\text{infinite disjunction}} \quad (\text{in } \mathcal{V})$$

Then $\psi(x_1,\dots,x_n) \leftrightarrow$ finite subdisjunction (in $\mathcal{V}$)

Proof: $\leftarrow$ for any subdisjunction

$\longrightarrow$ similar idea as $\mathcal{V}$=ab.gr. doesn't have DPC

add constants $a_1,\dots,a_n$ to the signature

$$\left.\begin{array}{l} \text{axioms for } \mathcal{V} \\ \psi(a_1,\dots,a_n) \\ \neg\psi_1(a_1,\dots,a_n) \\ \vdots \end{array}\right\} \text{no model} \Rightarrow \text{some finite subset has no model} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} \text{axioms for } \mathcal{V} \\ \psi(a_1,\dots,a_n) \\ \neg\psi_{m_1}(a_1,\dots,a_n) \\ \vdots \\ \neg\psi_{m_k}(a_1,\dots,a_n) \end{array}\right\} \text{no model} \Rightarrow \longrightarrow$$

UA 2 $\boxed{6.7}$

Theorem: $\mathcal{V}$ has DPC & finitely many SIs, all finite $\Rightarrow$ $\mathcal{V}$ is finitely based

Prf: • $\mathcal{V}$ has DPC witnessed by conservative $\varphi(x,y,a,b)$

$SI(\mathcal{V}) = \{\underline{A}_1, \dots, \underline{A}_k\}$ (up to iso)

• consider $\alpha_1 \wedge (\alpha_2 \to \alpha_3)$

where $\alpha_1 \equiv$ "$\forall a,b$ $\varphi\text{-}Cg^{\underline{A}}(a,b)$ is a congruence of $\underline{A}$ containing (a,b)"

$\forall a,b,x$ $\varphi(x,x,a,b)$ "$\varphi\text{-}Cg(a,b)$ is reflexive"

$\wedge$ $\forall a,b,x,y$ $\varphi(x,y,a,b) \to \varphi(y,x,a,b)$ "$\varphi\text{-}Cg(a,b)$ is symmetric"

$\wedge$ "$\varphi\text{-}Cg(a,b)$ is transitive"

$\wedge$ $\forall a,b$ $\varphi(a,b,a,b)$

$\wedge$ "$\varphi\text{-}Cg(a,b)$ compatible with operations"

$\alpha_2 \equiv$ "$\underline{A}$ is SI according to φ"

$\exists a \neq b$ $\forall x \neq y$ $\varphi(a,b,x,y)$

$\alpha_3 \equiv$ "$\underline{A} \in \{\underline{A}_1, \dots, \underline{A}_k\}$"

should be clear

• $\mathcal{V} \models \alpha_1 \wedge (\alpha_2 \to \alpha_3) \overset{\text{completeness}}{\Rightarrow} Id(\mathcal{V}) \vdash \alpha_1 \wedge (\alpha_2 \to \alpha_3) \Rightarrow$

$\Rightarrow \exists \mathcal{E} \subseteq_{fin} Id(\mathcal{V})$ such that $\mathcal{E} \vdash \alpha_1 \wedge (\alpha_2 \to \alpha_3)$.

• $\mathcal{W} = Mod(\mathcal{E})$, clearly $\mathcal{V} \subseteq \mathcal{W}$

• $\mathcal{W} \overset{?}{\subseteq} \mathcal{V}$ Enough $\underline{A} \in \mathcal{W}$ is SI $\Rightarrow \underline{A} \in \{\underline{A}_1, \dots, \underline{A}_k\}$

$a,b \in A$ $\underline{A} \models \alpha_1 \dashrightarrow \varphi\text{-}Cg^{\underline{A}}(a,b)$ is congruence containing (a,b)

φ is conservative $\dashrightarrow \varphi\text{-}Cg^{\underline{A}}(a,b) \subseteq Cg^{\underline{A}}(a,b)$

$\Big\}$ $Cg^{\underline{A}}(a,b) = \varphi\text{-}Cg^{\underline{A}}(a,b)$