UA2 $\boxed{5.1}$

# (IN)FINITE BASES

**Def.** Equational base for a variety $\mathcal{V}$ = a set $\mathcal{E}$ of identities such that $\mathcal{V} = \mathrm{Mod}(\mathcal{E})$

**Recall** $\forall$variety has an equational base

**Def.** $\mathcal{V}$ is finitely based, if it has a finite eq. base
($\Leftrightarrow$ $\exists$ finite $\mathcal{E}$ such that $\mathcal{V} \models s \approx t$ iff $\mathcal{E} \vdash s \approx t$)

$\underline{A}$ is finitely based, if $HSP(\underline{A})$ is

($\Leftrightarrow$ $\exists$ finite $\mathcal{E}$ such that $\underline{A} \models s \approx t$ iff $\mathcal{E} \vdash s \approx t$)

**Remark** in Mod-Id finitely based varieties $\leftrightarrow$ compact equational theories

**Examples**

- the variety of all groups
- — " — all rings
- — " — all lattices

$\Big\}$ finitely based

- — " — all vector spaces over $\mathbb{R}$ -- not finitely based

BUT $\infty$-many operation symbols ... cheating

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Assume: signature is finite!

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$\boxed{\text{Proposition}}$ $\mathcal{V}$ finitely based $\iff$ $\mathcal{V}$ can be finitely axiomatized using 1st order logic (1 sentence enough)

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Proof: $\Rightarrow$ triv.

$\Leftarrow$ • $\mathcal{E}$ some equational base for $\mathcal{V}$

• $\varphi$ sentence axiomatizing $\mathcal{V}$
(ie. $\underline{A} \in \mathcal{V} \iff \underline{A} \models \varphi$)

• $\mathcal{E} \models \mathcal{V} \Rightarrow \mathcal{E} \vdash \varphi$

$\exists$ formal proof of $\varphi$ from $\mathcal{E}$ (more general $\vdash$ than previously)

completeness thm. for 1st order logic

$\Rightarrow \exists$ finite $\mathcal{F} \subseteq \mathcal{E}$ such that $\mathcal{F} \vdash \varphi$

• $\mathcal{V} = Mod(\varphi) = Mod(\mathcal{F})$

$\supseteq$ $Mod(\mathcal{F}) \supseteq Mod(\mathcal{E}) = \mathcal{V}$

$\mathcal{F} \subseteq \mathcal{E}$

$\subseteq$ since $\mathcal{F} \vdash \varphi$

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$\boxed{\text{Main focus}}$: finite algebras

$\boxed{\text{History}}$

- not every finite algebra is finitely based [Lyndon '54]  ↙ 7-element example
  - 6-element semigroup [Perkins '69]

$$\left( \left\{ \begin{pmatrix} 0&0\\0&0 \end{pmatrix}, \begin{pmatrix} 1&0\\0&0 \end{pmatrix}, \begin{pmatrix} 0&1\\0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\1&0 \end{pmatrix}, \begin{pmatrix} 0&0\\0&1 \end{pmatrix}, \begin{pmatrix} 1&0\\0&1 \end{pmatrix} \right\} ; \cdot \right)$$

matrix mul.

  - 3-element grupoid [Murski '65]

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 |
| 2 | 0 | 2 | 2 |

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- some finitely based algebras
  - $\forall$ 2-element algebra [Lyndon '51]
  - $\forall$ finite group [Oates & Powell '65]

    Ex: $HSP(S_3) = Mod(\text{group axioms}, x^6 \approx 1, (x^{-1}y^{-1}xy)^3 = 1, x^2y^2 \approx y^2x^2)$

    how to show – syntactically (scary)
    – using SI's (not easy either)

    but not $\forall$ pointed group [Bryant]
  - $\forall$ finite ring [Kruse, Lvov 73]
  - $\forall$ finite commutative semigroup [Perkins '69]
  - $\forall$ finite lattice [McKenzie '70]

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$\boxed{5.4}$

## Which finite algebras are finitely based?

**Theorem [McKenzie '96]** The problem

INPUT: finite $\underline{A}$
OUTPUT: is it finitely based?

is undecidable

"Tarski's finite basis problem"

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"Park's conjecture" [76] $\underline{A}$ finite, and $\{\underline{B} \in HSP(\underline{A});\ \underline{B}\ SI\}$ is finite set of finite algebras. Then $\underline{A}$ is finitely based

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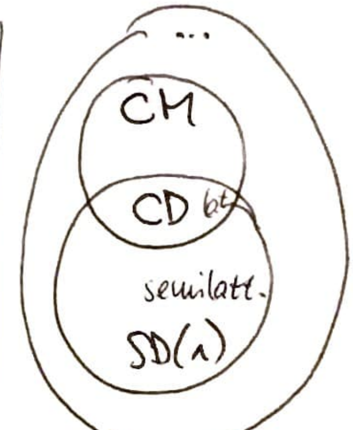
True if $HSP(\underline{A})$ is CD [Baker '72]

CM [McKenzie '87]

"meet semidistributive" SD($\wedge$) [Willard '00]

... [Willard, Kearnes, Szendrei '13]

87 EUR for counterexample!

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"0" $\underline{A} = (A; f_1, \dots, f_k)$, $g \in Clo(\underline{A})$. Then

$\underline{A}$ fin. based $\Leftrightarrow$ $(A; f_1, \dots, f_k, g)$ fin. based

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"0" $Clo(\underline{A}) = Clo(\underline{B})$. Then

$\underline{A}$ fin. based ~~iff~~ $\Leftrightarrow$ $\underline{B}$ fin. based

$\rightsquigarrow$ clone property!

$\boxed{\text{Theorem}}$ [Birhoff'35] $\underline{A}$ finite @

(1) $\underline{A}$ is finitely based

(2) $\exists n$ $\underline{A}$ has a base consisting of identities over $\{x_1, \dots, x_n\}$

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(1) $\Rightarrow$ (2) trivial

(2) $\Rightarrow$ (1) show that there is a "finite base of identities over $\{x_1, \dots, x_n\}$" for each finite $\underline{A}$

i.e. finite $\mathcal{E}$ over $\lambda$ such that $s \approx t$ (over $\{x_1, \dots x_n\}$) $\Rightarrow \mathcal{E} \vdash s \approx t$

- $C := F_{\underline{A}}(\{x_1, \dots, x_n\}) = F(\{x_1, \dots, x_n\})/\lambda$
- $C$ finite $\left(\underline{C} \simeq \underline{\mathbf{Clo}}_n \underline{A}\right)$
- for each $c \in C$ take $c^* \in C/\lambda$ so that $(x_i/\lambda)^* = x_i$ ~~$(x_i/\lambda)^* \neq x_i$~~
- $\mathcal{E} := \{ f(c_1^*, \dots, c_k^*) \approx [f(c_1, \dots, c_k)]^*;$
  $f \in \Sigma,\ ar(f) = k,\ c_1^*, \dots, c_k \in C \}$
  - by induction of height show $\mathcal{E} \vdash s \approx (s/\lambda)^*$
  - $s \approx t \Rightarrow s/\lambda = t/\lambda \Rightarrow \mathcal{E} \vdash s \approx (s/\lambda)^* = (t/\lambda)^* \approx t$

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Ex. $\underline{A} = (\{0,1\}, \wedge)$, $n = 2$, $F_{\underline{A}}(\{x,y\}) = \{x/\lambda, y/\lambda, x \wedge y/\lambda\}$

the stars

| | | |
|---|---|---|
| $\mathcal{E}$: $x \wedge x \approx x$ | $y \wedge x \approx x \wedge y$ | $(x \wedge y) \wedge x \approx x \wedge y$ |
| $x \wedge y \approx x \wedge y$ | $y \wedge y \approx y$ | $(x \wedge y) \wedge y \approx x \wedge y$ |
| $x \wedge (x \wedge y) \approx x \wedge y$ | $y \wedge (x \wedge y) \approx x \wedge y$ | $(x \wedge y) \wedge (x \wedge y) \approx x \wedge y$ |

(it is not a base for HSP($\underline{A}$))
need $n = 3$

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# A non-finitely based algebra [Park '80]

$\underline{A} = (\{0,1,2,\bot\}, \circ)$

| $\circ$ | 0 | 1 | 2 | $\bot$ |
|---|---|---|---|---|
| 0 | 0 | 1 | $\bot$ | $\bot$ |
| 1 | 1 | 1 | 2 | $\bot$ |
| 2 | $\bot$ | 2 | 2 | $\bot$ |
| $\bot$ | $\bot$ | $\bot$ | $\bot$ | $\bot$ |

more generally construction of an algebra

$\mathbb{D}$ ... digraph

$\underline{A} = (D \dot\cup \{\bot\}, \circ)$

$u \cdot v = v \cdot u = v$ if $u \to v$

$u \cdot v = \bot$ if $u \not\to v$ and $v \not\to u$


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- commutative, idempotent, not associative

$x_1 x_2 \dots x_n := ((((x_1 x_2) x_3) \dots ) x_n$

- for ↯ assume $\mathcal{E}$ finite base
- say all identities over $\{x_1, \dots, x_{n-1}\}$
- $s := x_1 x_2 \dots x_n x_1 x_2 \dots x_n$

  $t := x_2 x_3 \dots x_n x_1 x_2 \dots x_n x_1$

- will show $\underline{A} \models s \approx t$ but $\mathcal{E} \not\models s \approx t$ ↯

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$$\underline{A}: \quad \begin{tikzcd} 0 \arrow[r] \arrow[loop above] & 1 \arrow[r] \arrow[loop above] & 2 \arrow[loop above] \end{tikzcd}$$

$$s = x_1 x_2 \dots x_n x_1 x_2 \dots x_n$$
$$t = x_2 x_3 \dots x_n x_1 x_2 \dots x_n x_1$$

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$\boxed{\underline{A} \models s \approx t}$

- Pick $a_1, \dots, a_n \in A$
- $s^{\underline{A}}(a_1, \dots, a_n) \stackrel{?}{=} t^{\underline{A}}(a_1, \dots, a_n)$
  - $\exists i\ a_i = \perp$ ✓ (both LHS=RHS$=\perp$)
  - $\{0,2\} \subseteq \{a_1, \dots, a_n\}$ ✓
    LHS = RHS $= \perp$
  - $\{a_1, \dots, a_n\} \subseteq \{0,1\}$ ✓
    since $(\{0,1\}, \cdot|_{\{0,1\}})$ 2-element semilattice (commutative, associative)
  - $\{a_1, \dots, a_n\} \subseteq \{1,2\}$ ✓
    $(\{1,2\}, \cdot|_{\{1,2\}})$ —— " ——

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$\underline{A}$: $0 \circlearrowleft \longrightarrow 1 \circlearrowleft \longrightarrow 2 \circlearrowleft$

$s = x_1 x_2 \dots x_n x_1 x_2 \dots x_n$

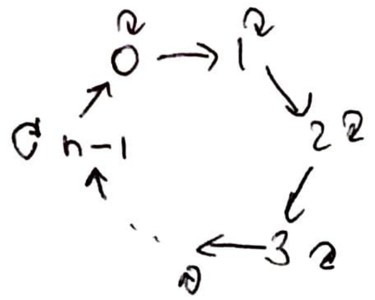
$t = x_2 x_3 \dots x_n x_1 \dots x_n x_1$

$\mathcal{E} \subseteq \{$identities over $\{x_1, \dots, x_{n-1}\}$ satisfied by $\underline{A}\}$

$\boxed{\mathcal{E} \not\models s \approx t}$ Find $\underline{B}$ such that (a) $\underline{B} \models \mathcal{E}$

(b) $\underline{B} \not\models s \approx t$

$\underline{B}$:



$$\left.\begin{array}{l} s^{\underline{B}}(0,1,\dots,n-1) = n-1 \\ t^{\underline{B}}(0,1,\dots,n-1) = 0 \end{array}\right\} (b) \checkmark$$

(a) for ↯ assume

- $p \approx q$ in $\mathcal{E}$
- $p^{\underline{B}}(b_1, \dots, b_{n-1}) \neq q^{\underline{B}}(b_1, \dots, b_{n-1})$

- some element of $\{0,1,\dots,n-1\}$ not among the $b_i$
- symmetry $\to$ WLOG assume $0 \notin \{b_1, \dots, b_{n-1}\}$
- $C := B \setminus \{0\} \leq \underline{B}$ $\quad \underline{C}$: $1 \circlearrowleft \longrightarrow 2 \circlearrowleft \longrightarrow \cdots \longrightarrow n-1 \circlearrowleft$
- $C \not\models p \approx q$ (because $p^{\underline{C}}(b_1, \dots, b_{n-1}) \neq q^{\underline{C}}(b_1, \dots, b_{n-1})$)
- But $\underline{C} \in HSP(\underline{A})$ ↯
  - $D \in S(\underline{A}^{n-3})$

$$\begin{array}{lcl} D = \{(0,\dots,0,0) & \overset{\theta}{\longmapsto} & 1 \\ (0,\dots,0,1) & \longmapsto & 2 \\ (0,\dots,0,1,2) & & \\ (0,\dots,0,1,2,2) & & \\ \vdots & & \\ (2,2,\dots,2) & \longmapsto & n-1 \\ \text{conteing } \perp\} & \longmapsto & \perp \end{array}$$

- $\theta : \underline{D} \to \underline{C}$