UA2 $\boxed{4.1}$

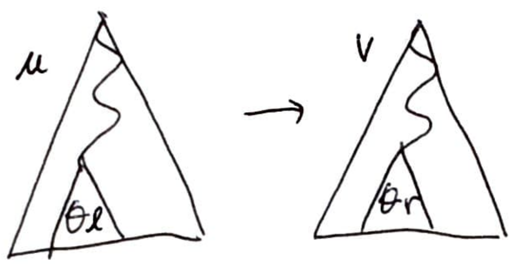
$\boxed{\text{RECAP}}$

$\mathcal{E}$ ... set of identities (ordered)

$D(\mathcal{E})$ vertices: terms (over $\{x_1, x_2, \dots\}$)

$u \to v$: $u \approx v$ is an immediate consequence of $l \approx r \in \mathcal{E}$, i.e.

$v = u(a: \theta l \to \theta r)$ for $\theta \in \mathrm{End}(\underline{F})$



- $\mathcal{E} \models s \approx t \iff \mathcal{E} \vdash s \approx t \overset{def}{\iff} s \leftrightarrow^* t$

- If $D(\mathcal{E})$ convergent, i.e,
  - finitely terminating
  - normal: $s \leftrightarrow^* t \Rightarrow \exists u \quad s \xrightarrow{*} u \xleftarrow{*} t$

  then
  - $\forall t \ \exists!$ terminal $NF(t)$ s.t. $t \longrightarrow^* NF(t)$
  - $s \leftrightarrow^* t$ iff $NF(s) = NF(t)$

TODAY: how to ensure
- finite termination
- normality

UA2 $\boxed{4.2}$

FINITE TERMINATION

Finitely terminating?

$\times$ $\mathcal{E} = \{x \approx xx\}$, $\mathcal{E} = \{xy \approx yx\}$

$\checkmark$ $\mathcal{E} = \{f(f(x)) \approx f(x)\}$ $\mathcal{E} = \{xx \approx x\}$

$\times!$ $\mathcal{E} = \{xx \cdot y \approx yy\}$ $(xx \cdot xx \to xx \cdot xx)$

Def. A strict ordering $<$ on F is a <u>reduction order</u> if

- $\nexists$ no infinite sequence $t_0 > t_1 > t_2 > \dots$
- $s > t \Rightarrow \theta(s) > \theta(t)$ for every $\theta \in End(\underline{F})$
- $s > t \Rightarrow \forall u, a$ $u[a] = s \Rightarrow u > u(a : s \to t)$

If $\mathcal{E}$ is <u>compatible</u> with a reduction order, then $D(\mathcal{E})$ is finitely terminating

$\forall l \approx r \in \mathcal{E}$ $l > r$

<u>Examples</u> (too simple)

$\checkmark$ $\Sigma = \{f\}$ $s > t \overset{def}{\equiv} \#f\text{'s in } s > \#f\text{'s in } t$

$\times$ $\Sigma = \{f, +\}$

$\checkmark$ Σ without constants $s > t \overset{def}{\equiv} \forall \text{var. } x$ $\#x\text{'s in } s \geq \#x\text{'s in } t$
and at lest 1 strict

$\times!$ $s > t \overset{def}{\equiv} \#\text{variables in } s > \#\text{variables in } t$

(see $xx \cdot y \approx yy$)

UA2 $\boxed{4.3}$

NORMALITY

Theorem: D finitely terminating digraph. @

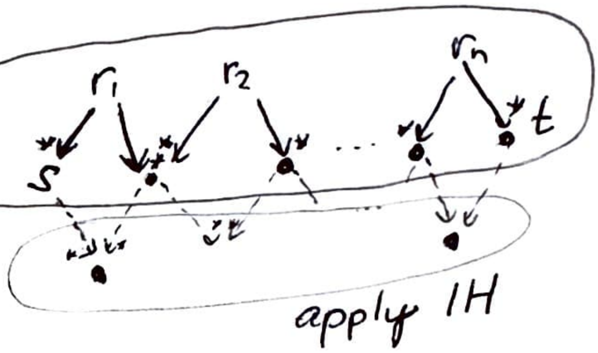
(i) D is normal $\quad s \leftrightarrow^* t \Rightarrow \exists u \quad s \to^* u \leftarrow^* t$

(ii) D is <u>confluent</u> $\quad r \to^* s,\ r \to^* t \Rightarrow \exists u \quad s \to^* u,\ t \to^* u$

(iii) D is <u>locally confluent</u> $\quad r \to s,\ r \to t \Rightarrow \exists u \quad s \to^* u,\ t \to^* u$

Proof: (i) ⇒ (ii) ⇒ (iii) trivial

(ii) ⇒ (i) $s \leftrightarrow^* t \iff \exists r_1, \dots, r_n$



induction on n

(iii) ⇒ (ii)

P := vertices that have directed path to ≥2 terminal vert.

- $P = \emptyset$ ✓ (by finite termination)
- $P \neq \emptyset$ take $r \in P$ "terminal in P", i.e. s.t. $\forall t\ r \to t \Rightarrow t \notin P$ (exists by finite termination)

def. of P • take s, t terminals $\quad r \to^* s,\ r \to^* t \qquad$ b $r \neq s,\ r \neq t$

• consider $s_1, t_1 \quad r \to s_1 \to^* s,\ r \to t_1 \to^* t$

local confluency • take terminal u $\quad r \to s_1 \to^* s,\ s_1 \to^* u,\ t_1 \to^* u,\ r \to t_1 \to^* t$

↯ to r terminal in P since $s \neq u$ or $u \neq t$

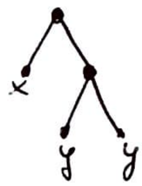
UA2 $\boxed{4.4}$

Example $\mathcal{E} = \{ x \cdot yy \approx xy \}$

$D(\mathcal{E})$ is not locally confluent

$[(xy)\cdot((uu \cdot uu))]\cdot y \xrightarrow{\text{apply on}} (xy\cdot(uu\cdot u))\cdot y$ terminal

$[(xy)\cdot((uu \cdot uu))]\cdot y \xrightarrow{\text{apply on } \square} (xy\cdot uu)\cdot y \to (xy\cdot u)\cdot y$ terminal

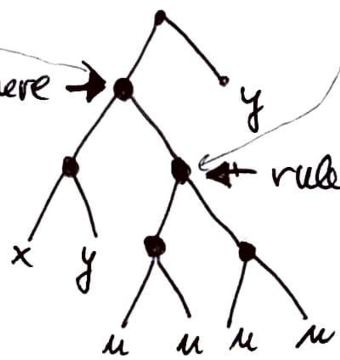
rule A  $\to$ ..

$\shortparallel$

rule B

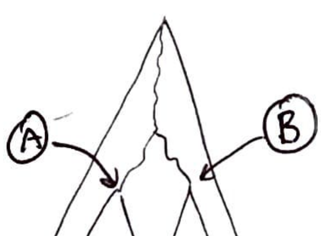
What is the problem?

rule A apply here

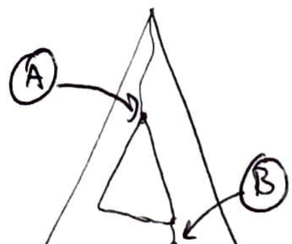


rule B apply here

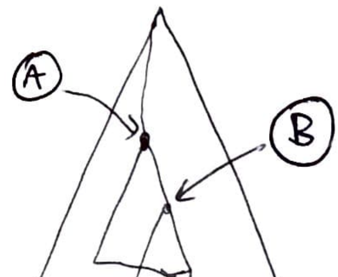
B is applied inside A



no problem



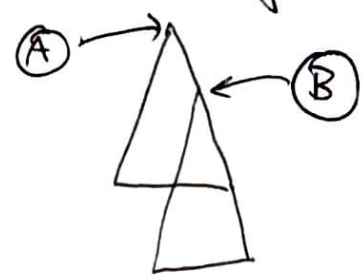
no problem



may be a problem

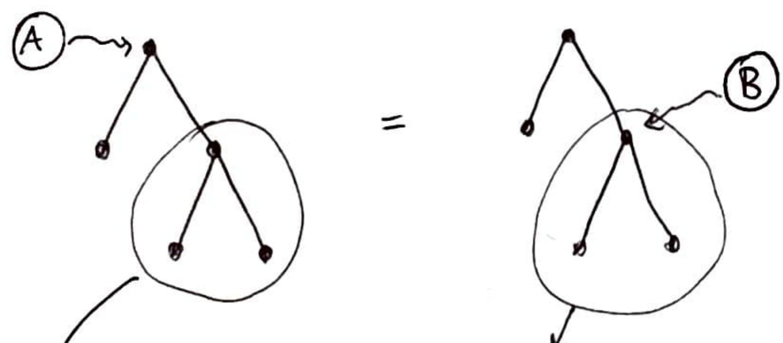
UA2 $\boxed{4.5}$

Critical pair .... "minimal instance" (or "most general") of the overlapping application problem



A $l_1 \approx r_1 = x \cdot yy \approx xy$

B $l_2 \approx r_2 = x \cdot yy \approx xy$



needs to be of the form

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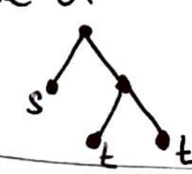
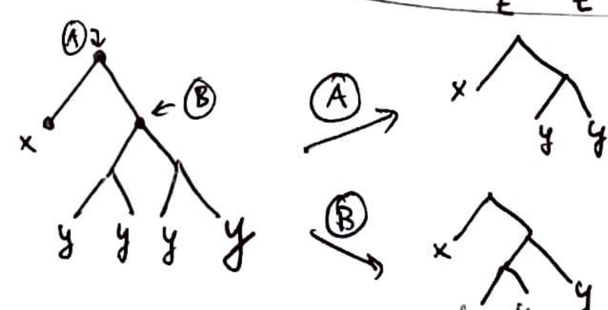


"most general" way



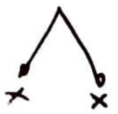
critical pair

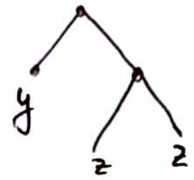
- more complex terms than x, y ... not general enough
- $x = y$ ... not general enough

UA2 $\boxed{4.5}$

MOST GENERAL UNIFIER

<u>Def.</u> $\theta \in \mathrm{End}(\underline{F})$ is a <u>unifier</u> of $s, t \in F$ if $\theta(s) = \theta(t)$

<u>Example</u> $s =$ 

$t =$ 

unifiers eg.

① θ: $x \mapsto xx$, $y \mapsto xx$, $z \mapsto x$, $w \mapsto x$, $v \mapsto x$, $\vdots$

② θ: $x \mapsto xx$, $y \mapsto xx$, $z \mapsto x$, $w \mapsto w$, $u \mapsto u$, $\vdots$

③ θ: $x \mapsto (xy)(xy)$, $y \mapsto (xy)(xy)$, $z \mapsto xy$, $w \mapsto w$, $\vdots$

<u>Def.</u> θ is a <u>most general unifier</u> of s, t, $MGU(s,t)$, if $\theta(s) = \theta(t)$ and $\forall$ unifier ω $\quad \omega = \alpha\theta$ for some $\alpha \in \mathrm{End}(\underline{F})$

<u>Example</u> ② is $MGU(s,t)$, ① & ③ are not

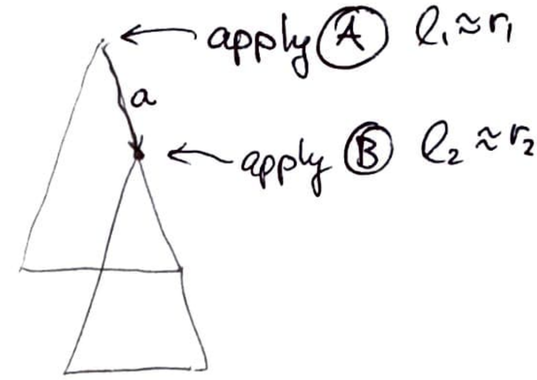
$\boxed{\text{theorem}}$ $\forall s, t$ — no unifier exists

— $\exists\, MGU(s,t)$, unique up to variable renaming

Proof: not required

4.6

CRITICAL PAIR



<u>Def.</u> Let

- $l_1 \approx r_1$, $l_2' \approx r_2' \in \mathcal{E}$
- $l_2 \approx r_2$ obtained from $l_2' \approx r_2'$ by renaming variables so that $l_1 \approx r_1$, $l_2 \approx r_2$ have disjoint sets of variables
- a address, $l' := l_1[a]$, $l' \neq$ variable (can be $a = \phi$)
- $\theta = MGU(l', l_2)$

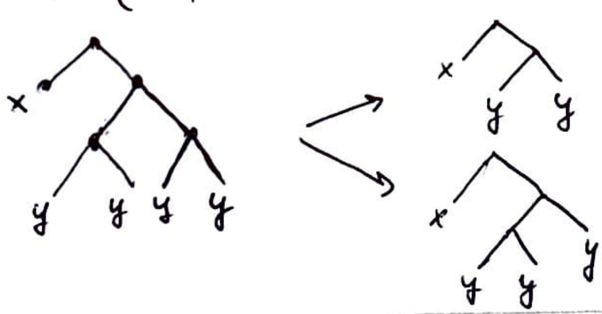
Then $(\theta r_1, \theta l_1(a : \theta l' \to \theta r_2))$ is a <u>critical pair</u>.

<u>Example</u> A $l_1 \approx r_1$ B $l_2 \approx r_2$ Q: why renaming?

$x \cdot yy \approx xy$ $u \cdot vv \approx uv$

θ: $y \mapsto yy$, $u \mapsto yy$, $v \mapsto y$, $x \mapsto x$, ⋮

- $a = 2$ $l' = l_1[2] = yy$
- $\theta = MGU(l', l_2') = MGU(yy, u \cdot vv) =$
- θl_1



θr_1

$\theta l_1(a : \theta l' \to \theta r_2')$

$\boxed{\text{Theorem}}$ $D(\mathcal{E})$ finitely terminating and has confluent critical pairs, then $D(\mathcal{E})$ locally confluent ($\Rightarrow$ convergent)

UA2 4.7

KNUTH-BENDIX ALGORITM

INPUT: $\mathcal{E}$ a set of identities

OUTPUT: equivalent $\hat{\mathcal{F}}$ which is convergent
(or failure, or works forever)

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parameter: $>$ reduction order

- order identities in $\mathcal{E}$ so that $l > r$ for each $l \approx r \in \mathcal{E}$ (or fail if impossible)
- $\mathcal{C} :=$ all critical pairs in $\mathcal{E}$
- while $\mathcal{C} \neq \emptyset$
  - choose $(s, t) \in \mathcal{C}$ and remove from $\mathcal{C}$
  - compute terminal vertices s_0, t_0 such that
    $s \to^* s_0 \qquad t \to^* t_0$
  - if $s_0 \neq t_0$
    - order (or fail)
    - add to $\mathcal{E}$
    - update $\mathcal{C}$