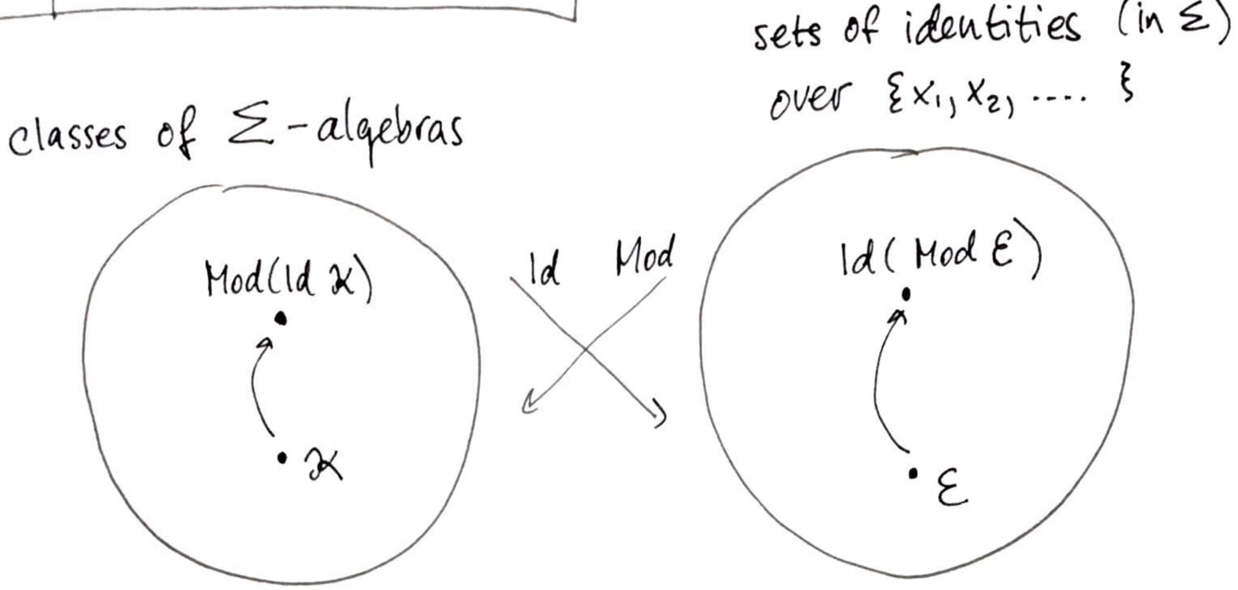
UA2 $\boxed{3.1}$

Equational theories



closure of $\mathcal{K} \overset{\text{def}}{=}$ all algs satisfying all identities satisfied by $\mathcal{K}$

$\overset{\text{Birkhoff}}{=}$ smallest variety containing $\mathcal{K}$

$= HSP(\mathcal{K})$

closure of $\mathcal{E} \overset{\text{def}}{=}$ all identities that semantically follow from $\mathcal{E}$

$(\mathcal{E} \models s \approx t$ if $\forall \underline{A}\ \ \underline{A} \models \mathcal{E} \Rightarrow \underline{A} \models s \approx t)$

= the smallest equational theory containing $\mathcal{E}$

= all identities that syntactically follow from $\mathcal{E}$

Today: this ↗

Next:
- how to decide $\mathcal{E} \models s \approx t$?
- given $\mathcal{V}$ is there a finite $\mathcal{E}$ such that $\mathcal{V} = Mod(\mathcal{E})$?

UA2 $\boxed{3.2}$

Equational theory

<u>Def.</u> <u>Equational theory</u> = fully invariant congruence $\approx$ of $\underline{F} := F_\Sigma(x_1, x_2, \dots\dots)$

$\uparrow$ absolutely free algebra

for each $\theta \in \mathrm{End}(\underline{F})$, for every $s \approx t$, $\theta(s) \approx \theta(t)$

(O) endomorphism of $\underline{F}$ = substitution

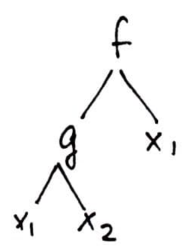
– why? endomorphisms determined by values on variables

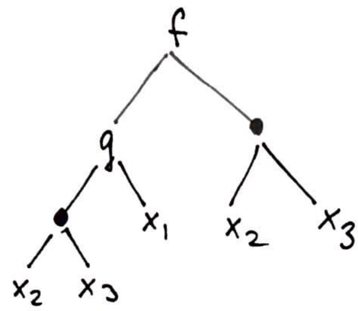
e.g. if

$$x_1 \xmapsto{\theta} x_2 \cdot x_3$$
$$x_2 \longmapsto x_1$$
$$x_3 \longmapsto x_3$$
$$\vdots$$

then

$$f(g(x_1, x_2), x_1) \overset{\theta}{\mapsto} f(g(x_2 \cdot x_3, x_1), x_2 \cdot x_3)$$





(O) $\mathrm{Id}(\mathcal{K})$ is always an equational theory

equivalence ✓

congruence e.g. $s_1 \approx s_2$, $t_1 \approx t_2 \overset{✓}{\Longrightarrow} s_1 \cdot t_1 \approx s_2 \cdot t_2$

e.g. $xy \cdot z \approx x \cdot yz$, $u \approx u \Rightarrow (xy \cdot z) \cdot u \approx (x \cdot yz) \cdot u$

(use equations inside terms)

fully invariant $s \approx t \overset{✓}{\Longrightarrow} \theta(s) \approx \theta(t)$ (substitute)

e.g. $xy \cdot z \approx x \cdot yz$ $\quad x \xmapsto{\theta} uv$, $y \longmapsto y$, $z \longmapsto y$ $\quad \Rightarrow (uv)y \cdot y \approx uv \cdot yy$

UA2 $\boxed{3.3}$

ie. smallest one containg $\mathcal{E}$

$\boxed{\text{Theorem}}$

$\mathcal{E}$ .. set of identities. Then $\mathrm{Id}(\mathrm{Mod}\,\mathcal{E})$ = equational theory generated by $\mathcal{E}$

$\Rightarrow$ closed elements on the right (in Mod-Id correspondence) = equational theories

Proof: $\supseteq$ ✓ (from ☺)

$\subseteq$ $s \approx t \in \mathrm{Id}(\mathrm{Mod}\,\mathcal{E}) \overset{?}{\Rightarrow} s\,\alpha\,t$ — congruence of $\underline{F}$ on RHS

- $\underline{A} := \underline{F}/\alpha$ (correct, α is a congruence of $\underline{F}$)

- $\underline{A} \in \mathrm{Mod}(\mathcal{E})$ $\quad u \approx v \in \mathcal{E} \overset{?}{\Rightarrow} \underline{A} \models u \approx v$

want for each $m: X \to F/\alpha$ $\quad \hat{m}(u) = \hat{m}(v)$ $(\hat{m}: \underline{F} \to \underline{F}/\alpha)$

pick $X \to F$, $x \mapsto \in m(x)$

use $\theta: \underline{F} \to \underline{F}$ extending this & full invariance

$\Rightarrow \underline{A} \models s \approx t$

apply this for $m: X \to F/\alpha$, $x \mapsto x/\alpha$

we have $\hat{m}(s) = \hat{m}(t)$

$\hat{m}(s) = s/\alpha$, $\hat{m}(t) = t/\alpha$ $\quad \Rightarrow s\,\alpha\,t$

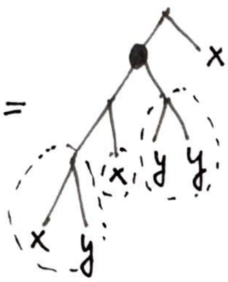
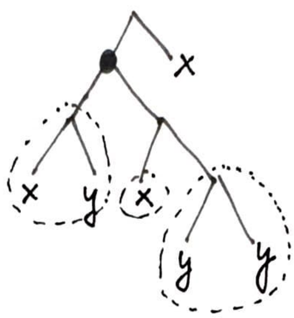
UA 2 $\boxed{3.4}$

Immediate consequence

want: $u \approx v$ is an immediate consequence of $l \approx r$ if

$u =$  $v =$  for some $\theta \in End(E)$

e.g. $l \approx r$ is $(xy)z \approx x(yz)$

$u =$  $v =$ 

θ

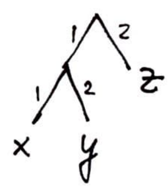
$x \mapsto xy$

$y \mapsto x$

$z \mapsto yy$

"applying substituted $l \approx r$ inside a term (once)"

Terminology

e.g. $t =$ 

<u>address</u> ... sequence of natural numbers (possibly $\emptyset$)

<u>valid address</u> for a term — e.g. ✓ $11, 12, 2, \emptyset$ ✗ $3, 21, \ldots$

<u>$t[a]$</u> where t is a term and a a valid address for t — e.g. $t[\emptyset] = t$, $t[1] =$  $t[12] = y$

<u>$t(a: u \to v)$</u> defined if $t[a] = u$
= term obtained by replacing u by v at address a

e.g. $t[1: xy \to z(xx)]$
$= [z(xx)]z$ 

$t[12: y \to x]$
$= (xx)z$

e.g. $t[1: x \to y]$ undefined

UA 2 $\boxed{3.5}$

$\boxed{\vdash}$

viewpoint: v is obtained by rewriting u using the rule $\ell \to r$
$\rightsquigarrow$ term rewriting

$\boxed{\text{Def.}}$ $u \approx v$ is an <u>immediate consequence</u> of $\ell \approx r$ if
$\exists a \quad \exists \theta \in End(\underline{F})$ such that $v = u(a: \theta\ell \to \theta r)$

$\boxed{\text{Note}}$: not symmetric

$\boxed{\text{Def.}}$ $\mathcal{E} \vdash s \approx t$ if $\exists s = u_1, u_2, \dots, u_n = t$ such that
$\forall_i \quad u_i \approx u_{i+1}$ or $u_{i+1} \approx u_i$ is an immediate consequence of an equation in $\mathcal{E}$

Theorem (equational completness theorem)

$$\mathcal{E} \models s \approx t \quad \text{iff} \quad \mathcal{E} \vdash s \approx t$$

Proof: ($\Leftarrow$) clear

($\Rightarrow$) because of theorem ↑ enough to show

$\{(s,t);\ \mathcal{E} \vdash s \approx t\}$ = equational theory generated by $\mathcal{E}$

$\subseteq$ ✓

$\supseteq$
- equivalence
- congruence
- fully invariant

} Exercise

UA2 $\boxed{3.6}$

$\boxed{\text{Term rewriting}}$ approach to deciding $\mathcal{E} \vdash s \approx t$

$\mathcal{E}$ ... set of identities ("rewriting system")

want: INPUT: s, t
OUTPUT: $\mathcal{E} \vdash s \approx t$?

Digraph $D(\mathcal{E})$ [vertices: terms over $\{x_1, x_2, \dots\}$
$u \to v$: (u,v) is an immediate consequence of $(l,r) \in \mathcal{E}$

$\to^*$ reflexive transitive closure of $\to$
($s \to^* t$ if $\exists\, s \to \to \to \dots \to t$)

$\leftrightarrow^*$ reflexive symmetric transitive closure of $\to$
= connectivity
($s \leftrightarrow^* t$ if $\exists\, s \to \leftarrow \leftarrow \to \leftarrow \leftarrow \leftarrow \to \dots t$)

We know: $\mathcal{E} \models s \approx t$ iff $\mathcal{E} \vdash s \approx t$ iff $s \leftrightarrow^* t$

If $\to$ is very nice (convergent),
then we can decide $s \leftrightarrow^* t$

UA 2

3.7

Convergent digraphs

<u>Def.</u> Digraph D is

- <u>finitely terminating</u> if there is no infinite directed path from a vertex
  (no $\bullet \longrightarrow \longrightarrow \longrightarrow \cdots\cdots$)
- <u>normal</u> if $\forall s,t \quad s \longleftrightarrow^* t \implies \exists u \quad s \longrightarrow^* u, \; t \longrightarrow^* u$ ($\Leftarrow$ automatic)
- <u>convergent</u> if it is finitely terminating and normal

<u>terminal vertex</u> of D = no arrow from it

D is convergent iff $\forall x \; \exists!$ terminal vertex $NF(x)$ such that $x \longrightarrow^* NF(x)$

"normal form of x"

and then $s \longleftrightarrow^* t$ iff $NF(s) = NF(t)$

If $D(\mathcal{E})$ is convergent

- we "can decide" whether $\mathcal{E} \vdash s \approx t$ $(\Leftrightarrow NF(s)=NF(t$

some assumptions needed

- and have normal form of terms

Knuth-Bendix algorithm

INPUT: $\mathcal{E}$

OUTPUT: $\mathcal{F}$ convergent, $Mod(\mathcal{E}) = Mod(\mathcal{F})$

- fails
- works forever
- succeeds ☺