UA2 $\boxed{2.1}$

$\boxed{\text{Recap}}$

affine = polynomially equivalent to a module

abelian = $\forall t \in Clo(\underline{A})\ \forall \bar{x}, \bar{y} \in A^k\ \forall \bar{u}, \bar{v} \in A^\ell$

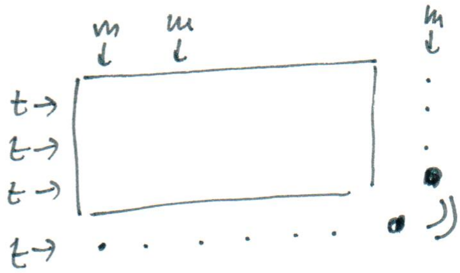
$$t(\bar{x}, \bar{u}) = t(\bar{x}, \bar{v}) \Rightarrow t(\bar{y}, \bar{u}) = t(\bar{y}, \bar{v})$$

Ex. • affine

• unary

Equivalent def. : $k = 1$

$\underline{A}$ affine $\Rightarrow$ abelian and has exactly one Mal'cev polynomial, which is central



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abelian + Mal'cev $\Rightarrow$ affine

UA2 $\boxed{2.2}$

$\boxed{\text{When abelian} \Rightarrow \text{affine ?}}$

✓ if $HSP(\underline{A})$ congruence permutable
$\Leftrightarrow$ $\underline{A}$ has a Mal'cev term operation

✓ if $HSP(\underline{A})$ congruence distributive
$\Leftarrow$ $\underline{A}$ has a majority term operation

– only singleton algebras are abelian

✓ if $HSP(\underline{A})$ congruence modular (generalizes both ↑)
[Herrmann '77]

✓ if $\underline{A}$ is finite and satisfies any nontrivial Mal'cev condition
(generalizes all above for <u>finite</u> algebras)
[Hobby, McKenzie '80s - Tame congruence theory
easy proof via absorption]

<u>Example</u> $(\mathbb{Q}; \frac{x+y}{2})$
- abelian
- satisfies a nontrivial idempotent Mal'cev condition $\left(\begin{array}{l} x*y = y*x \\ x*x = x \end{array}\right)$
- no Mal'cev polynomial operation (use $\leq$)

UA2 $\boxed{2.3}$

Relational characterization of abelianess

<u>Proposition</u> $\underline{A}$ algebra.

(i) $\underline{A}$ is abelian

(ii) Δ $(= \{(a,a);\ a \in A\})$ is a block of a congruence of $\underline{A}^2$

Proof: <u>Note</u>: (ii) $\Leftrightarrow$

Δ is a block of $Cg_{\underline{A}^2}(\Delta^2)$

<u>Recall</u> (?) $\underline{B}$ algebra, $X \subseteq B \times B$

$Cg_{\underline{B}}(X) =$ transitive symmetric closure of $\{(p(a), p(b));\ (a,b) \in X,\ p$ unary polynomial op.$\}$

$$Cg_{\underline{A}^2}(\Delta^2) = \text{trans. sym. cl. of } \left\{\left(p\binom{a}{a}, p\binom{b}{b}\right);\ a,b \in A,\ p \text{ unary polynomial op. of } \underline{A}^2\right\}$$

$$= \text{—''—} \left\{\left(t\left(\binom{a}{a}, \binom{c_1}{d_1}, \dots, \binom{c_n}{d_n}\right), t\left(\binom{b}{b}, \binom{c_1}{d_1}, \dots, \binom{c_n}{d_n}\right)\right);\ a, b, \bar{c}, \bar{d} \in A^{\dots},\ t \in Clo(\underline{A})\right\}$$

Δ is a block of $\uparrow$ iff

$$t\left(\binom{a}{a}, \binom{c_1}{d_1}, \dots\right) \in \Delta \Leftrightarrow t\left(\binom{b}{b}, \binom{c_1}{d_1}, \dots\right)$$

iff (i) (def. of abelianess with $k=1$)

$\square$

UA 2 $\boxed{2.4}$

$\boxed{\text{Example}}$

Let $\underline{A} = (\{0,1\}; \ldots)$, $\underline{A}$ has a Mal'cev polynomial operation (or ....), $R := \{000, 011, 101, 110\} \leq \underline{A}^3$.

Claim: $\underline{A}$ is affine (ie $\underline{A}$ poly. eq. to $(\mathbb{Z}_2, +)$)

- Note: $R(xyz)$ iff $x+y+z = 0 \pmod 2$
- By proposition, enough to find congruence $\sim$ of $\underline{A}^2$ whose block is $\{(0,0), (1,1)\} = \Delta$

$\to$ Find $\sim \leq (\underline{A}^2)^2 \cong \underline{A}^4$ whose block is Δ

- How to produce subpowers (= compatible rel's)?
  $\to$ pp-definitions

$\to$ pp-define from R a relation $S \leq \underline{A}^4$ so that when viewed as $\sim \subseteq (A^2)^2$ it is an equivalence whose one block is Δ

$S(x, y, x', y') \overset{def}{\equiv}$

UA2 $\boxed{2.5}$

A taste of commutator theory

<u>Def.</u> $\alpha, \beta, \gamma \in \operatorname{Con} \underline{A}$, <u>$\alpha$ centralizes β modulo δ</u>, written $C(\alpha, \beta; \delta)$ if

$x_1 \,\alpha\, y_1,\ x_2 \,\alpha\, y_2, \dots$

$$\forall t \in \operatorname{Clo}(\underline{A})\ \ \forall \bar{x}, \bar{y} \in A^k \quad \bar{x} \ \alpha\ \bar{y}$$
$$\forall \bar{u}, \bar{v} \in A^l \quad \bar{u}\ \beta\ \bar{v}$$

$$t(\bar{x}, \bar{u})\ \delta\ t(\bar{x}, \bar{v}) \Rightarrow t(\bar{y}, \bar{u})\ \delta\ t(\bar{y}, \bar{v})$$

<u>Remarks</u>

- $\underline{A}$ abelian if $C(1_A, 1_A, 0_A)$

no need to memorize

- <u>def</u>: $\alpha \leq \beta$ in $\operatorname{Con} \underline{A}$. β <u>abelian</u> over α if $C(\beta, \beta; \alpha)$
- <u>def</u> $\alpha, \beta \in \operatorname{Con} \underline{A}$. <u>commutator</u> of α, β is the smallest δ such that $C(\alpha, \beta; \delta)$
  - well defined
  - $[\alpha, \beta] \leq \alpha \wedge \beta$
  - in general $[\alpha, \beta] \neq [\beta, \alpha]$ (but $=$ in CM)
- <u>def.</u> $\underline{A}$ <u>solvable</u> ~~iff~~ if ...

  <u>nilpotent</u> if .....

.... in groups it matches the standard concepts

UA 2 $\boxed{2.6}$

Proposition $\underline{G}$ group, $\alpha, \beta, \delta \in \mathrm{Con}\,\underline{G}$,
A, B, D the corresponding normal subgroups
(ie $A = 1/\alpha$, $B = 1/\beta$, $D = 1/\delta$)

Then $C(\alpha, \beta; \delta)$ iff $\forall a \in A\ \forall b \in B\ \ ab\ \delta\ ba$

($\Leftrightarrow aba^{-1}b^{-1} \in D$)

Proof: • WLOG $\delta = 1$ $\left(\begin{array}{l} \text{as } C(\alpha,\beta;\delta) \text{ in } \underline{A} \text{ iff} \\ C(\alpha/\delta, \beta/\delta; 0) \text{ in } \underline{A}/\delta \end{array}\right)$

$\Rightarrow$ $\underset{x_1}{1} \cdot \underset{u_1}{b} \cdot \underset{x_2}{1^{-1}} \cdot \underset{u_2}{b^{-1}} = \underset{y_1}{1} \cdot \underset{v_1}{1} \cdot \underset{y_2}{1^{-1}} \cdot \underset{v_2}{1^{-1}}$ Note $\bar{u}\ \beta\ \bar{v}$, $\bar{x}\ \alpha\ \bar{y}$

$\Rightarrow a \cdot b \cdot a^{-1} \cdot b^{-1} = a \cdot 1 \cdot a^{-1} \cdot 1^{-1} = 1$

$\Leftarrow$ we assume $\forall a \in A\ \forall b \in B\ \ ab = ba\ \ (aba^{-1}b^{-1} = 1)$

$\overset{!!!}{0}$ 1) $\forall \overline{w_1}, \overline{w_2}\ \ \overline{w_1}\ \beta\ \overline{w_2}\ \ \forall r \in \mathrm{Clo}(\underline{A})\ \ r(\overline{w_1})\, r(\overline{w_2})^{-1} \in B$

$\overset{!!!}{0}$ 2) $\forall w_1, w_2\ \ w_1\ \alpha\ w_2\ \ \forall b \in B\ \ w_1^{-1} b w_1 = w_2^{-1} b w_2$

Prf: $\underbrace{w_2 w_1^{-1}}_{\in A}\ \underset{\overset{\Uparrow}{B}}{b}\ w_1 w_2^{-1} \overset{?}{=} b$

$\| \qquad\qquad \|$

$b\, w_2 w_1^{-1} w_1 w_2^{-1}$

UA2 $\boxed{2.7}$

- $x \,\alpha\, y$, $\bar{u} \,\beta\, \bar{v}$, $t \in \mathrm{Clo}(\underline{A})$

want $t(x, \bar{u}) = t(x, \bar{v}) \Rightarrow t(y, \bar{u}) = t(y, \bar{v})$

will prove $t(x, \bar{u})\, t(x, \bar{v})^{-1} = t(y, \bar{u})\, t(y, \bar{v})$

- $t(x_1, \dots, x_n) = z_1 \cdot z_2 \cdot \dots \cdot z_m$ where each z_i is $x_1^{\pm 1}, x_2^{\pm 1}, \dots,$ or, $x_n^{\pm 1}$

induction on m

$m = 1$
- $t(x_1, \dots x_n) = x_1$ $\quad x x^{-1} \overset{?}{=} y y^{-1}$ ✓ $\quad$ similarly x_1^{-1}
- $t(x_1, \dots x_n) = x_2$ $\quad u_1 v_1^{-1} \overset{?}{=} u_1 v_1^{-1}$ ✓ $\quad x_2^{-1}, x_3^{\pm 1} \dots$

ind. step:
- $t(x_1, \dots, x_n) = x_1\, s(x_1, \dots x_n)$ $\quad$ similarly x_1^{-1}

$$\begin{aligned}
LHS &= x_1\, s(x, \bar{u}) \big(x_1\, s(x, \bar{v})\big)^{-1} = \\
&= x_1\, s(x, \bar{u})\, s(x, \bar{v})^{-1}\, x_1^{-1} = \\
&\overset{IH}{=} x_1\, \underbrace{s(y, \bar{u})\, s(y, \bar{v})^{-1}}_{\text{in } B \text{ by } \ddot{\odot} 1}\, x_1^{-1} = \\
&\overset{\ddot{\odot} 2}{=} y_1\, s(y, \bar{u})\, s(y, \bar{v})^{-1}\, y_1^{-1} = RHS
\end{aligned}$$

- $t(x_1, \dots, x_n) = x_2\, s(x_1, \dots, x_n)$ $\quad$ similarly $x_2^{-1}, x_3^{\pm 1}, \dots$

$$\begin{aligned}
LHS &= u_1\, s(x, \bar{u})\, s(x, \bar{v})^{-1}\, v_1^{-1} \\
&\overset{IH}{=} u_1\, s(y, \bar{u})\, s(y, \bar{v})^{-1}\, v_1^{-1} \\
&= RHS
\end{aligned}$$