UA2 $\boxed{1.1}$

Outline

- abelianess (group theory → general algs)

  useful concept in many advanced theories

- equational theories
  - term rewriting, Knuth-Bendix algorithm
  - (in)finitely based varieties

- Taylor algebras/clones
  - CSP
  - Taylor algebras = satisfying some nontrivial Mal'cev condition
  - absorption theory

UA2 | 1.2

Abelianess

- modules are nice, quite well understood
- which algebras are "essentially modules"?

  how to recognize it?

optimally $Clo(\underline{A}) = Clo(\underline{M})$ for some module $\underline{M}$

a bit weaker.. $\underline{A}$ is affine

$\boxed{\text{Def.}}$ <u>Polynomial operation</u> of $\underline{A}$: operation of the form $f(x_1, \ldots, x_n) = t(x_1, \ldots, x_n, a_1, \ldots, a_k)$

where $a_i \in A$, $t \in Clo_{n+k}(\underline{A})$

$\boxed{\text{Ex.}}$

- $\underline{A} = (\{0,1\}; *)$ $f(x,y) = ((x * 0) * 1) * y$
- polynomial operations of an R-module $\underline{M}$ of the form $f(x_1, \ldots, x_n) = \sum a_i x_i + c$

  where $a_i \in R$, $c \in M$

- all polynomial operations (of arity ≥ 1)

  $= Clo(\underline{A} + \text{constant operations})$

- polynomial operations of a ring = polynomials

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Def. $\underline{A}, \underline{B}$ <u>polynomially equivalent</u> if $A=B$ and they have the same polynomial operations

i.e. $\mathrm{Clo}(\underline{A} + \text{constants}) = \mathrm{Clo}(\underline{B} + \text{constants})$

<u>recall</u> term equivalent: $\mathrm{Clo}(\underline{A}) = \mathrm{Clo}(\underline{B})$

Def. $\underline{A}$ is <u>affine</u> if it is polynomially equivalent to an R-module (for some ring R)

Ex.
- $(\mathbb{Z}_p; +_{\mathrm{mod}\ p})$ is affine
- abelian group is affine $(R = \mathbb{Z})$
- $(\mathbb{Z}_p^k;$ operations of the form $A_1x_1 + A_2x_2 + \dots + A_nx_n$

  $k \times k$ matrices over $\mathbb{Z}_p$

  $(R = \mathbb{Z}_p^{k\times k})$

Smith | Gumm 70s : crucial insights

① $\underline{A}$ affine $\Rightarrow$ $\underline{A}$ has a unique Mal'cev polynomial op. which is <u>central</u>

② $\underline{A}$ affine $\Rightarrow$ $\underline{A}$ is <u>abelian</u>

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Def. m (ternary) is <u>central</u> if for every basic operation f of $\underline{A}$ "$m(f,f,f) = f(m,m,\ldots,m)$"

i.e. $m(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n), f(z_1,\ldots,z_n)) = f(m(x_1,y_1,z_1), \ldots, m(x_n,y_n,z_n))$

$$\begin{array}{c|cccc|c}
 & \downarrow m & \downarrow m & & \downarrow m & \downarrow m \\
\hline
\xrightarrow{f} & x_1 & x_2 & \cdots & x_n & \bullet \\
\xrightarrow{f} & y_1 & y_2 & \cdots & y_n & \bullet \\
\xrightarrow{f} & z_1 & z_2 & \cdots & z_n & \bullet \\
\hline
\xrightarrow{f} & \bullet & \bullet & & \bullet & \bullet \ \bullet
\end{array}$$

equal

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(0) $\underline{A}$ affine $\Rightarrow$ $\underline{A}$ has a unique Mal'cev polynomial op. which is central

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Prf: • uniqueness

say $m(x,y,z) = ax + by + cz + d$ is Mal'cev ($a,b,c \in R$, $d \in M$)

Mal'cev identities:

$x = m(x,y,y) = ax + (b+c)y + d$

$ = m(y,y,x) = (a+b)y + cx + d$

- plug in $y=0$, $x=0$ $\rightsquigarrow$ (a bit of work) $d=0$, $x = ax = cx$
- $x = ax + by + cy + d = x + by + y \Rightarrow by = -y$

$\Rightarrow m(x,y,z) = x - y + z$

• existence ✓ is Mal'cev

• centrality: easy check

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Def. $\underline{A}$ is abelian if $\forall t$ term/polynomial operation

$\forall \underbrace{\bar{x}, \bar{y}}, \underbrace{\bar{u}, \bar{v}}$ tuples of elements of A

the same length

$$\underbrace{t(\bar{x}, \bar{u}) = t(\bar{x}, \bar{v}) \Rightarrow t(\bar{y}, \bar{u}) = t(\bar{y}, \bar{v})}_{\text{"the term condition"}}$$

Remarks
- $\bar{x}, \bar{y}$ of length 1 $\rightsquigarrow$ the same concept ☺
- $\bar{u}, \bar{v}$ — " — $\rightsquigarrow$ not the same concept
- what does it say for binary operations?

☺ $\underline{A}$ affine $\Rightarrow$ $\underline{A}$ abelian

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Prf: $t(w_1, \dots, w_k, z_1, \dots, z_\ell) = \sum a_i w_i + \sum b_i z_i + c$

..

Examples

- unary algebras – abelian
- semilattices – never (unless trivial)
- groups: abelian $\Leftrightarrow$ abelian
- rings: abelian $\Leftrightarrow$ $\forall x, y \; x \cdot y = 0$

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$\boxed{\text{Fundamental Theorem on abelian algebras}}$

(a) $\underline{A}$ affine $\Leftrightarrow$ $\underline{A}$ has a Mal'cev polynomial operation and $\underline{A}$ is abelian

(b) Assume $\underline{A}$ has a Mal'cev polynomial operation m

(1) $\underline{A}$ is abelian

(2) $\underline{A}$ is affine

(3) m is central

Pf: enough to prove (b), $(2) \Rightarrow (1)$, $(2) \Rightarrow (3)$ ✓

$(3) \Rightarrow (1)$ assume $t(x, \bar{u}) = t(x, \bar{v})$

use

$$\boxed{\begin{array}{cccccc} y & u_1 & u_2 & \cdots & \cdots & u_n \\ x & u_1 & u_2 & \cdots & & u_n \\ x & v_1 & v_2 & \cdots & & v_n \end{array}}$$

$(1) \Rightarrow (2)$

if we knew it was true, how would we reconstruct $+, -, 0, R$ from $\underline{A}$

....

....

...

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$\boxed{\text{Mal'cev + abelian} \Rightarrow \text{affine}}$

pick $0 \in A$ arbitrarily

$$\left.\begin{aligned} x + y &:= m(x, 0, y) \\ -x &:= m(0, x, 0) \end{aligned}\right\} \begin{array}{c} (A, +, -, 0) \\ \text{abelian group} \\ \text{(needs to be checked)} \end{array}$$

$$R := \{\text{all unary polynomial operations } f \text{ with } f(0) = 0\}$$

$$\left.\begin{aligned} f \cdot g &:= f \circ g \\ (f+g)(x) &:= f(x) + g(x) \\ &\ldots \end{aligned}\right\} \text{ring } R$$

$r.x := r(x)$ .. R-module structure $\underline{M}$

clear: all operations of $\underline{M}$ are polynomial op. of $\underline{A}$

todo: —"— $\underline{A}$ —"— of $\underline{M}$

induction on arity; $n = 1$ ✓

consider $t(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - f(0, x_2, \ldots x_n) - f(x_1, 0, \ldots 0) + f(0, 0, \ldots 0)$

$$t(0, x_2, \ldots, x_n) = t(0, 0, \ldots, 0) \overset{\text{term c.}}{\Longrightarrow} t(x_1, \ldots x_n) = t(x_1, 0, \ldots 0) = 0$$

$$\Rightarrow f(x_1, \ldots, x_n) = f(0, x_2, \ldots x_n) + f(x_1, 0, \ldots, 0) - f(0, \ldots, 0)$$