

1) Let $\underline{L} = (L, \wedge, \vee)$ be a distributive lattice. Show that

$$(c, d) \in \text{Con}_{\underline{L}}((a, b)) \Leftrightarrow \begin{cases} c \wedge (a \vee b) = d \wedge (a \vee b) \text{ and} \\ c \vee (a \wedge b) = d \vee (a \wedge b) \end{cases}$$

Hence the variety of distributive lattices has DPC.

2) Show that the variety of seuilattices has DPC.

3) Let A be a finite relational structure. Show that $\exists B$ such that

(i) \exists homomorphisms $A \rightarrow B$
 $B \rightarrow A$

(ii) $\text{End}(B) = \text{Aut}(B)$

Further B is unique up to isomorphism.

(B is called the core of A)

Solutions

1) First observe that $\forall \theta \in \text{Con } L$: (also in gen. lattices)

$$(x, y) \in \theta \Leftrightarrow (x \wedge y, x \vee y) \in \theta$$

$$\left[\begin{array}{l} \text{Proof } \Rightarrow (x, y) \in \theta \\ \quad \underline{(x \wedge x) \in \theta} \\ \rightarrow (x, x \wedge y) \in \theta \\ \quad (x, x \vee y) \in \theta \end{array} \right. \Leftrightarrow \left[\begin{array}{l} (y, y) \in \theta \\ \quad \underline{(x \wedge x) \in \theta} \\ \rightarrow (x \wedge y, x \vee y) \in \theta \\ \rightarrow (x \wedge y, \underbrace{x \wedge (x \vee y)}_{=x}) \in \theta \\ \quad (x \wedge y, \underbrace{y \wedge (x \vee y)}_{=y}) \in \theta \end{array} \right.$$

Hence $\mathcal{C}_{g_L}((a, b)) = \mathcal{C}_{g_L}((a \wedge b, a \vee b))$ and it is enough to prove:

$$\forall a \leq b: (c, d) \in \mathcal{C}_{g_L}((a, b)) \Leftrightarrow \begin{cases} c \wedge a = d \wedge a & \& \\ c \vee b = d \vee b \end{cases}$$

$$\text{Let } \theta_{ab} := \{ (x, y) \in L^2 \mid x \wedge a = y \wedge a, x \vee b = y \vee b \}$$

Claim $\theta_{ab} \in \text{Con } L$ and $(a, b) \in \theta_{ab}$

$$\left[\begin{array}{l} \text{Proof} \\ \bullet \theta_{ab} \text{ equivalence relation } \checkmark \\ \bullet \text{ if } (c, d) \in \theta_{ab} \Rightarrow \begin{array}{l} c \wedge a = d \wedge a \\ (e, f) \in \theta_{ab} \Rightarrow \underline{e \wedge a = f \wedge a} \end{array} \quad \begin{array}{l} c \vee b = d \vee b \\ \underline{e \vee b = f \vee b} \end{array} \\ \rightarrow (c \wedge e) \wedge a = (d \wedge f) \wedge a \quad (c \vee b) \wedge (e \vee b) = (d \vee b) \wedge (f \vee b) \\ \Rightarrow (c \wedge e, d \wedge f) \in \theta_{ab} \quad \begin{array}{l} \parallel \\ \underline{(c \wedge e) \vee b} \end{array} \quad \begin{array}{l} \wedge (f \vee b) \\ \parallel \\ \underline{(d \wedge f) \vee b} \end{array} \\ \text{symmetrically } (c \vee e, d \vee f) \in \theta_{ab} \\ \bullet a \wedge a = a \wedge b \quad b \vee b = a \vee b \Rightarrow (a, b) \in \theta_{ab} \end{array} \right.$$

$$\Rightarrow \mathcal{C}_{g_L}((a, b)) \subseteq \theta_{ab}$$

To show $\Theta_{ab} \subseteq \mathcal{C}_{\mathcal{L}}((a,b)) =: \sim$

Let $(c,d) \in \Theta_{ab}$

then

$$c \sim c$$

$$a \sim b$$

$$c \wedge a \sim c \wedge b$$

\Leftrightarrow

$$d \wedge a \sim d \wedge b$$

$$\Leftrightarrow c \wedge b \sim d \wedge b$$

$$c = c \wedge (c \vee b) \stackrel{\textcircled{x}}{=} c \wedge (d \vee b) = (c \wedge d) \vee (c \wedge b)$$

$$d = d \wedge (d \vee b) \stackrel{\textcircled{y}}{=} d \wedge (c \vee b) = (c \wedge d) \vee (d \wedge b) \stackrel{\textcircled{z}}{=}$$

so $c \sim d$.

Thus $\mathcal{C}_{\mathcal{L}}((a,b)) = \Theta_{ab}$.

□

2) For a semilattice $\underline{L} = (L, \wedge)$, and $a, b \in L$
let us define

$$\theta_{ab} = \{(x, x) \mid x \in L\} \cup \{(x, y) \in L^2 \mid \begin{array}{l} (x \leq a \text{ or } x \leq b) \text{ and} \\ (y \leq a \text{ or } y \leq b) \text{ and} \\ x \wedge a \wedge b = y \wedge a \wedge b \end{array}\}$$

Note that $\psi(ab \times y) \Leftrightarrow \theta_{ab}(x, y)$ has a first-order definition.

Claim: $\text{Cg}_{\underline{L}}((a, b)) = \theta_{ab}$

1) \subseteq θ_{ab} is an equivalence relation \checkmark (easy)

2) $\theta_{ab} \in \text{Con } \underline{L}$: Yes, since $x \wedge (a \wedge b) = y \wedge (a \wedge b)$
 $\vee \wedge (a \wedge b) = v \wedge (a \wedge b)$

3) $(a, b) \in \theta_{ab} \checkmark \Rightarrow (x \wedge u) \wedge (a \wedge b) = (y \wedge v) \wedge (a \wedge b)$

since $a \wedge a \wedge b = a \wedge b = b \wedge a \wedge b$

\supseteq Let $(c, d) \in \theta_{ab}$
If $c = d \Rightarrow (c, d) \in \text{Cg}_{\underline{L}}((a, b))$

else, wlog $c \leq a$
 $d \leq b$:

$$c \sim c$$

$$a \sim b$$

$$c = c \wedge a \sim c \wedge b = c \wedge (a \wedge b) \stackrel{*}{=} d \wedge (a \wedge b) \stackrel{***}{=} d \wedge a$$

$$\Rightarrow (c, d) \in \text{Cg}_{\underline{L}}((a, b))$$

\square

$$d \wedge b = d$$

3) We prove the existence of B by induction on $|A|$, if $A = (A; R_1, R_2 \dots R_k)$.

• For $|A| = 1$

$$\text{End } A = \text{Aut } A = \{\text{id}\} \text{ so } B := A$$

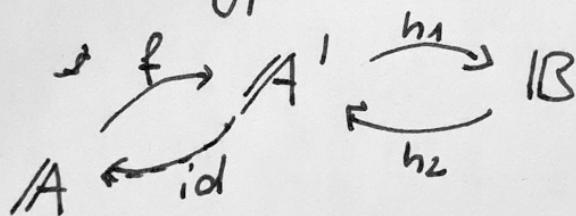
• For general $|A|$,

→ if $\text{End } A = \text{Aut } A$, then $B := A$

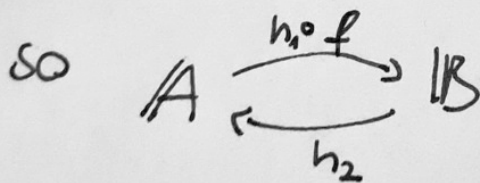
→ else $\exists f \in \text{End } A \setminus \text{Aut } A$

then $A' := f(A)$, $|A'| = |f(A)| < |A|$

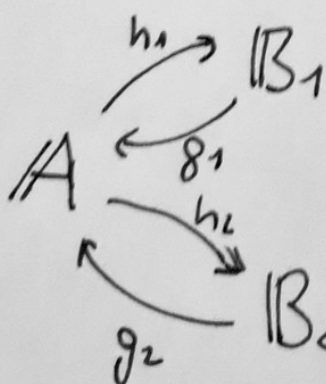
By induction hypothesis $\exists B$: monomorphisms h_1, h_2 ,



$$\text{End } B = \text{Aut } B$$



for uniqueness of B , assume there are two cores B_1, B_2 for given A :



then $\overline{h_1 \circ g_2 \circ h_2 \circ g_1} \in \text{End } B_1$
 $= \text{Aut } B_1$

$$\overline{h_2 \circ g_1 \circ h_1 \circ g_2} \in \text{Aut } B_2$$

\Rightarrow both $h_2 \circ g_1$ and $h_1 \circ g_2$ must be injective, and hence bijective.

as their compositions are ~~isom~~ automorphisms they are isom.