Univ. Alg. 2
Practicals
21.3.23

1) Let $\underline{L} = (L, \wedge, \vee)$ be a <u>distributive lattice</u>. Show that

$$(c,d) \in Cg_{\underline{L}}((a,b)) \iff \begin{cases} c \wedge (a \wedge b) = d \wedge (a \wedge b) \text{ and} \\ c \vee (a \vee b) = d \vee (a \vee b) \end{cases}$$

Hence the variety of distributive lattices has DPC.

2) Show that the variety of <u>semi</u>lattices has DPC.

3) Let $\mathbb{A}$ be a <u>finite</u> relational structure. Show that $\exists\, \mathbb{B}$ such that

(i) $\exists$ homomorphisms $\mathbb{A} \to \mathbb{B}$
$\mathbb{B} \to \mathbb{A}$

(ii) $End(\mathbb{B}) = Aut(\mathbb{B})$

Further $\mathbb{B}$ is unique up to isomorphism.

($\mathbb{B}$ is called the <u>core</u> of $\mathbb{A}$)

Solutions

1) First observe that $\forall \Theta \in \mathrm{Con}\,\underline{L}$: (also in gen. lattices)

$$(x,y)\in\Theta \Leftrightarrow (x\wedge y, x\vee y)\in\Theta$$

Proof "$\Rightarrow$" $(x,y)\in\Theta$, $(x,x)\in\Theta$ $\Rightarrow$ $(x, x\wedge y)\in\Theta$, $(x, x\vee y)\in\Theta$

"$\Leftarrow$" $(y,y)\in\Theta$, $(x,x)\in\Theta$, $(x\wedge y, x\vee y)\in\Theta$ $\Rightarrow$ $(x\wedge y, \underbrace{x\wedge(x\vee y)}_{=x})\in\Theta$, $(x\wedge y, \underbrace{y\wedge(x\vee y)}_{=y})\in\Theta$

Hence $Cg_L((a,b)) = Cg_L((a\wedge b, a\vee b))$ and it is enough to prove:

$$\forall \underline{a\leq b}: \quad (c,d)\in Cg_L((a,b)) \Leftrightarrow \begin{cases} c\wedge a = d\wedge a \ \& \\ c\vee b = d\vee b \end{cases}$$

Let $\Theta_{a,b} := \{(x,y)\in L^2 \mid x\wedge a = y\wedge a,\ x\vee b = y\vee b\}$

Claim $\Theta_{ab}\in \mathrm{Con}(\underline{L})$ and $(a,b)\in\Theta_{ab}$

Proof
- Θ_{ab} equivalence relation ✓
- If $(c,d)\in\Theta_{ab} \Rightarrow c\wedge a = d\wedge a$ $\quad c\vee b = d\vee b$
  $(e,f)\in\Theta_{ab} \Rightarrow e\wedge a = f\wedge a$ $\quad e\vee b = f\vee b$
  $\Rightarrow (c\wedge e)\wedge a = (d\wedge f)\wedge a$ $\quad (c\vee b)\wedge(e\vee b) = (d\vee b)\wedge(f\vee b)$, where $(c\vee b)\wedge(e\vee b) = (c\wedge e)\vee b$ and $(d\vee b)\wedge(f\vee b) = (d\wedge f)\vee b$

  $\Rightarrow (c\wedge e, d\wedge f)\in\Theta_{ab}$

  symmetrically $(c\vee e, d\vee f)\in\Theta_{ab}$
- $a\wedge a = a\wedge b$ $\quad b\vee b = a\vee b \Rightarrow (a,b)\in\Theta_{ab}$

$\Rightarrow Cg_L((a,b)) \subseteq \Theta_{ab}$

To show $\Theta_{ab} \subseteq Cg_L((a,b)) =: \sim$

Let $(c,d) \overset{\circledast}{\in} \Theta_{ab}$

then
$$\frac{\begin{array}{c} c \sim c \\ a \sim b \end{array}}{c \wedge a \sim c \wedge b}$$
$$\overset{\circledast}{\parallel}$$
$$d \wedge a \sim d \wedge b \quad \Rightarrow c \wedge b \overset{\circledast\circledast}{\sim} d \wedge b$$

$$c = c \wedge (c \vee b) \overset{\circledast}{=} c \wedge (d \vee b) = (c \wedge d) \vee (c \wedge b)$$
$$d = d \wedge (d \vee b) \overset{\circledast}{=} d \wedge (c \vee b) = (c \wedge d) \vee (d \wedge b)$$
$\wr \circledast\circledast$

So $c \sim d$.

Thus $Cg_L((a,b)) = \Theta_{ab}$.

$\square$

2) For a semilattice $\underline{L}=(L,\wedge)$, and $a,b\in L$ let us define

$$\theta_{ab}=\{(x,x)\mid x\in L\}\cup\{(x,y)\in L^2 \mid (x\le a \text{ or } \cancel{\ } \, x\le b) \text{ and } (y\le a \text{ or } y\le b) \text{ and } x\wedge a\wedge b = y\wedge a\wedge b\}$$

Note that $\varphi(abxy) \Leftrightarrow \theta_{ab}(x,y)$ has a first-ord definition.

<u>Claim</u>: $Cg_{\underline{L}}((a,b)) = \theta_{ab}$

„$\subseteq$" •) θ_{ab} is an equivalence relation ✓ (easy)

•) $\theta_{ab}\in \mathrm{Con}\,\underline{L}$: Yes, since $x\wedge(a\wedge b)=y\wedge(a\wedge b)$, $u\wedge(a\wedge b)=v\wedge(a\wedge b)$

$\Rightarrow (x\wedge u)\wedge(a\wedge b)=(y\wedge v)\wedge(a\wedge b)$

•) $(a,b)\in\theta_{ab}$ ✓

since $a\wedge a\wedge b = a\wedge b = b\wedge a\wedge b$

„$\supseteq$" Let $(c,d)\overset{*}{\in}\theta_{ab}$

If $c=d \Rightarrow (c,d)\in Cg_{\underline{L}}((a,b)) =: \sim$

Else, wlog $\left.\begin{matrix} c\le a \\ d\le b\end{matrix}\right]^{**}$:

$$\frac{\begin{matrix} c\sim c \\ a\sim b\end{matrix}}{}$$

$c \overset{**}{=} c\wedge a \sim c\wedge b = c\wedge(a\wedge b) \overset{*}{=} d\wedge(a\wedge b) \overset{**}{=} d\wedge a$

$\Rightarrow (c,d)\in Cg_{\underline{L}}((a,b))$ $\qquad d\wedge b = d$

3) We prove the <u>existence</u> of $\mathbb{B}$ by induction on $|A|$, if $\mathbb{A} = (A; R_1, R_2 .. R_k)$.

- For $|A| = 1$

  $\operatorname{End} \mathbb{A} = \operatorname{Aut} \mathbb{A} = \{id\}$ so $\mathbb{B} := \mathbb{A}$

- For general $|A|$,

  $\rightarrow$ if $\operatorname{End} \mathbb{A} = \operatorname{Aut} \mathbb{A}$, then $\mathbb{B} := \mathbb{A}$

  $\rightarrow$ else $\exists f \in \operatorname{End} \mathbb{A} \setminus \operatorname{Aut} \mathbb{A}$

  then $\mathbb{A}' := f(\mathbb{A})$, $|A'| = |f(A)| < |A|$

  By induction hypothesis $\exists \mathbb{B}$: homomorphisms h_1, h_2, $\operatorname{End} \mathbb{B} = \operatorname{Aut} \mathbb{B}$

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "f"] & \mathbb{A}' \arrow[l, bend left, "id"] \arrow[r, bend left, "h_1"] & \mathbb{B} \arrow[l, bend left, "h_2"] \end{tikzcd}$$

so
$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "h_1 \circ f"] & \mathbb{B} \arrow[l, bend left, "h_2"] \end{tikzcd}$$

For <u>uniqueness</u> of $\mathbb{B}$, assume there are two cores $\mathbb{B}_1, \mathbb{B}_2$ for given $\mathbb{A}$:

$$\begin{tikzcd} & \mathbb{B}_1 \arrow[dl, bend left, "g_1"] \\ \mathbb{A} \arrow[ur, bend left, "h_1"] \arrow[dr, bend left, "h_2"] & \\ & \mathbb{B}_2 \arrow[ul, bend left, "g_2"] \end{tikzcd}$$

then $\overline{h_1 \circ g_2} \circ \overline{h_2 \circ g_1} \in \operatorname{End} \mathbb{B}_1 = \operatorname{Aut} \mathbb{B}_1$

$\overline{h_2 \circ g_1} \circ \overline{h_1 \circ g_2} \in \operatorname{Aut} \mathbb{B}_2$

$\Rightarrow$ both $h_2 \circ g_1$ and $h_1 \circ g_2$ must be injective, and hence bijective.

as their compositions are ~~isomo~~ automorphisms they are <u>isom.</u>