UA I
5.1.24
Practicals 11

An algebra $\underline{A}$ is called <u>congruence distributive</u> (CD) if Con $\underline{A}$ is a distributive lattice.

A variety V is CD if all $\underline{A} \in V$ are CD.

We prove Jónssons characterization of CD varieties:

<u>Theorem (Jónsson 1967)</u> (Theorem 4.66 in Bergman)

Let V be a variety. Then TFAE:

(a) V is CD

(b) $\underline{F} := \underline{F}_V(\{x,y,z\})$ is CD.

(c) $\exists\, n \geq 1$, and ternary terms $p_0, p_1, \dots, p_n$ such that V satisfies

$\boxed{\text{Jónsson terms}}$

(i) $p_i(xyx) \approx x$ for all $0 \leq i \leq n$

(ii) $p_0(xyz) \approx x$ (iii) $p_n(xyz) \approx z$

(iv) $p_i(xxy) \approx p_{i+1}(xxy)$ for all even i

(v) $p_i(xyy) \approx p_{i+1}(xyy)$ for all odd i

1) Show that every variety with a majority is CD.
(* can you do it without using Jónsson's theorem?)
Conclude that lattices are CD.

2) recall that the condition $(\alpha \vee \beta) \wedge \gamma \leq (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$ ⊛ $\forall\, \alpha, \beta, \gamma \in \underline{L}$ is enough for $\underline{L}$ to be a distributive lattice.

<u>Proof of Jónsson's theorem</u>

3) (a) $\Rightarrow$ (b) trivially holds

4) For (b) $\Rightarrow$ (c), let $\alpha = Cg_{\underline{F}}(x,y)$, $\beta = Cg_{\underline{F}}(y,z)$, $\gamma = Cg_{\underline{F}}(x,y)$

From $(x,z) \in (\alpha \vee \beta) \wedge \gamma \leq (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$ it follows that $\exists\, u_0, u_1, u_2 \dots u_n \in F$ with

$$x = u_0 \ (\alpha \wedge \gamma)\ u_1\ (\beta \wedge \gamma)\, u_2 \dots (\beta \wedge \gamma)\, u_n = z$$

Show that $u_i = p_i^{\underline{F}}(xyz)$ gives us Jónsson terms p_i.

5)* Prove (c) $\Rightarrow$ (a).