20.10
UA 1
Practical 3

1) Prove that every complete lattice is <u>bounded</u>.

2) Find examples of lattices $\underline{L}$ with sublattice $\underline{S}$ such that:

   a) $\underline{L}$ is complete, $\underline{S}$ is not complete
   b) $\underline{L}$ is not complete, $\underline{S}$ is complete
   c) $\underline{L}, \underline{S}$ are complete, but $\underline{S}$ is <u>not</u> a <u>complete sublattice</u>

3) Let $\underline{L}$ be an algebraic lattice and $a, b \in L$ compact.

   a) Is $a \vee b$ compact?
   b) Is $a \wedge b$ compact?

4) Let C be a closure operator on a set A. Prove that L_C is closed under finite unions if and only if

$$C(X \cup Y) = C(X) \cup C(Y)$$

5) Let X be a set. For $U, V \subseteq X$ we define

$$(U, V) \in \Phi \Leftrightarrow U \cap V \neq \emptyset$$

we study the Galois connection induced by $\Phi \subseteq P(X) \times P(X)$.

   a) For $X = \{1,2,3,4\}$ compute the ~~closure~~ Galois closure $\mathcal{A}^{\to\leftarrow}$ for $\mathcal{A} = \{\{1\}, \{2,3\}\}$ and $\mathcal{A} = \{\{1,2\}, \{2,3\}\}$.

   b) Argue that $\mathcal{A}^{\to\leftarrow} = \mathcal{A}^{\leftarrow\to}$? What is a general property of Φ that implies this symmetry?

   c) Show that

$$\mathcal{A}^{\to\leftarrow} = \{U \in P(X) \mid \exists V \in \mathcal{A} : V \subseteq U\}.$$

(Hint: For general X, this requires the axiom of choice)

6) Let C be a closure operator on a set A. Find a Galois connection $\Phi \subseteq A \times P(A)$, such that

$$C(X) = X^{\to\leftarrow} \quad \text{for all } X \subseteq A.$$