NMAG405 - Universal Algebra 1 - winter term 2023/24

Homework 4

Deadline 22.12.23 12:20

1. (10 points) Let $\mathbf{R}$ be a fixed ring with unity, and V be the variety of $\mathbf{R}$-modules $(A, +, 0, -, (r)_{r \in R})$ (i.e. the elements of the ring correspond to the unary operations $r(x) = r \cdot x$). Show that the free algebra $\mathbf{F}_V(X)$ is isomorphic to the module $R^{(X)} = \{(u_x)_{x \in X} \in R^X \mid u_x \neq 0 \text{ for only finitely many } x \in X\}$. Use the definition of free algebras in the sense of universal algebra (i.e. not the definition from module/category theory).

2. (10 points) For two varieties of groups V, W we define the variety $V \cdot W = \{G \mid \exists N \trianglelefteq G, N \in V, G/N \in W\}$. Let $\mathcal{A}_2$ be the variety of Abelian groups satisfying $x^2 \approx 1$. Show that $\mathcal{A}_2 \cdot \mathcal{A}_2$ is the variety of groups that satisfy $(x^2y^2)^2 \approx 1$.

3. (10 points) Let A be a finite set. Show that the clone of *all* operations on A is already generated by all binary operations on A.