NMAG405 - Universal Algebra 1 - winter term 2023/24

Homework 3

Deadline 08.12.23 12:20

1. (10 points) Let $\mathbf{R}$ be a commutative ring such that for every non-zero element a also $a^n \neq 0$ holds for every $n > 1$ in $\mathbf{R}$ (such a ring is also called *reduced*).

   - Show that for every $a \in R \setminus \{0\}$ there is a prime ideal P_a with $a \notin P_a$ (Hint: Pick P_a as a maximal ideal that does not contain any power $a, a^2, a^3, \ldots$)
   - Prove that $\mathbf{R}$ is the subdirect product of integral domains.

2. (10 points) Let $\mathcal{V}$ be the variety of algebras $(A, \cdot, l, r)$ of type $(2, 1, 1)$ that satisfy the identities
$$l(x \cdot y) \approx x, \quad r(x \cdot y) \approx y, \quad l(x) \cdot r(x) \approx x.$$

   (a) Show that every non-trivial member of $\mathcal{V}$ is infinite.

   (b) Prove that, if $\mathbf{A} \in \mathcal{V}$ is generated by $\{a_1, a_2, \ldots, a_n\}$, then it is already generated by $\{(a_1 \cdot a_2), a_3, \ldots, a_n\}$

   (c) Prove that $\mathbf{F}_{\mathcal{V}}(n) = \mathbf{F}_{\mathcal{V}}(m)$ for all positive integers n, m.

3. (10 points) Let $\mathbf{A}$ be the semigroup given by the following multiplication table:

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

   Prove that the variety generated by $\mathbf{A}$ is exactly the variety of commutative semigroups satisfying $x^3 \approx x^4$.