NMAG405 - Universal Algebra 1 - winter term 2023/24

Homework 2

Deadline 24.11.23 12:20

1. (10 points) Determine all the subuniverses and congruences of $(\mathbb{N}, +, *)$ where

$$x * y = \begin{cases} 0 & \text{if } y = 0, 1 \\ x \bmod y & \text{else,} \end{cases}$$

   and describe the lattices $\mathrm{Sub}((\mathbb{N}, +, *))$ and $\mathrm{Con}((\mathbb{N}, +, *))$.

2. (10 points) Let $\mathbf{G} = (G, \cdot, ^{-1}, e)$ be a group. Prove that there is a lattice isomorphism between the lattice of normal subgroups of $\mathbf{G}$ and the lattice of congruences of $\mathbf{G}$.

3. (10 points) For a fixed prime p consider the algebra $\mathbf{A} = (\{0, 1, \ldots, p-1\}, m)$, where m is a ternary operation defined by $m(x, y, z) = x - y + z \bmod p$. Prove that for any n, R is a subuniverse of $\mathbf{A}^n$ if and only if R is empty or an affine subspace of $\mathbb{Z}_p^n$ (recall the definition of affine subspace from linear algebra).